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CWI Tract 50

Asymptotic analysis of a class of perturbed Korteweg-de Vries initial value problems

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PREFACE

This tract has been written at Eindhoven University of Technology. The research, however, was done at the State University of Utrecht.
The presentation is intended to be self-contained. Except for some familiarity with asymptotic expansions, no specific mathematical background is required.
The material covered is outlined in detail in Chapter 1. Chapter II can serve as an introduction to the Inverse Scattering method.
I wish to thank Prof. W. Eckhaus for his support during the preparation of this tract. I am also grateful to the Mathematics Department of Eindhoven University of Technology for giving me the opportunity to finish this tract.
My special thanks to Prof. M. Hazewinkel for his careful reading of the manuscript. He corrected many of my grammatical mistakes.
Finally, I thank Elsina Baselmans who typed the manuscript faster and more accurate than I thought was possible.

F. de Kerf
January 1988

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CHAPTER I
INTRODUCTION

1. Historical Introduction

In 1895, D.J. Korteweg and G. de Vries derived a model equation for the behaviour of long waves in shallow water, [KdV]. In dimensionless scaled variables this equation, the so-called *Korteweg-de Vries equation*, is:

(1.1) $$u_t - 6uu_x + u_{xxx} = 0 \ , \quad \text{(KdV)}.$$

This equation has *solitary wave* solutions:

(1.2) $$u(x,t) = 2a^2 \operatorname{sech}^2 a(x - 4a^2 t - x_0) \ .$$

In physical variables these solitary waves represent shallow water waves.

In 1965, N.J. Zabusky and M.D. Kruskal, [ZK], discovered the *soliton*. They studied the KdV because of its relevance to plasma physics, as well as, to the Fermi-Pasta-Ulam problem. An interesting account of the motivations of Zabusky and Kruskal for studying the KdV has been given by Kruskal, see [Kr]. They took two waves of type (1.2) with the smallest one in front as the initial condition for the KdV. By means of numerical integration they found that the larger solitary wave overtook the smaller one and came in front. The remarkable fact is that the only effect of their interaction was a change of phase, compared with the positions they would have had without mutual interaction. The larger solitary wave was shifted to the right, while the smaller one was shifted to the left. This particle-like behaviour inspired Zabusky and Kruskal to the name 'soliton'.
Nowadays it has become common use to use the term soliton for any solitary wave.

A great breakthrough came in 1967 when C.S. Gardner, J.M. Greene, M.D. Kruskal and R.M. Miura found a way to solve the KdV initial value problem analytically by means of the spectral transform technique, [GGKM 1].

Soon afterwards Lax put their method into a mathematical framework, that clearly indicated its generality (see Lax). The method became known as the Inverse Scattering Transform, (IST). A search was opened for nonlinear evolution equations that are solvable by IST. Following Calogero and Degasperis we will call such evolution equations S-*integrable*. As it turned out, there are large classes of S-integrable equations (for instance, see [AKNS], [Cd 1,2,3], [Lax], [ZS]).

Since 1967, much literature about the KdV, solitons and IST has appeared, especially the textbooks: [AS], [Cd 4], [DEGM], [EvH], [L], [NMPZ]. Moreover, the concept of the soliton and IST has spread out to other areas of mathematics, such as algebraic and differential geometry and functional and numerical analysis. Applications of the subject occur through the whole of physics. For more literature about IST and solitons, I refer to [Cd 4], in which a wealth of references has been given.

The first scientific description of the soliton as a natural phenomenon was given by J. Scott Russell in the first half of the nineteenth century, [SR]. While riding on horseback beside a channel, the boat he was observing suddenly stopped. Scott Russell noted that it set forth:
"........ a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminuition of speed. .... Its hight gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon ...."

2. Main goal of this research; Short description of the way by which this goal is achieved

When deriving a mathematical model that describes some physical phenomenon, 'small' terms are being neglected. Therefore it is important to investigate how such a model behaves under perturbations. A way of doing this is inserting new small terms in the model. This will be done for the KdV initial value problem.

We consider:

(1.3) $$\begin{cases} u_t - 6uu_x + u_{xxx} = \varepsilon f(u) \ , \\ u(x,0) = U(x) \end{cases}$$

where f(u) is some function of the real variables u and x-derivatives of u. An interesting perturbation is given by:

(1.4) $$f(u) = \frac{3}{2}u^2 u_x + \frac{5}{2} uu_{xxx} - \frac{23}{4} u_x u_{xx} + \frac{19}{40} u_{xxxxx} \ .$$

This expression is found when the KdV is derived for shallow water waves by means of an expansion in the small parameter ε, neglecting terms of order ε^2 and higher. ($\varepsilon \approx \frac{a}{h} \approx \frac{h^2}{\ell^2}$, where a is a typical wave-amplitude, ℓ a typical wave-length and h the depth of the water.) A derivation of this perturbed KdV (pKdV) is given in Appendix F.1.

The construction of a perturbation theory for the KdV or other non-linear S-integrable evolution equations is far from completed. Steps in this direction have been described in: [EvH], [J], [K], [KA], [KK], [KM], [KMcL], [KN], [KS], [LSO], [McLS], [N]. In this thesis a consistent perturbation theory for the problem (1.3) is presented which is based on the idea of applying the IST to (1.3). As a starting point, we take the formal perturbation procedure as outlined by W. Eckhaus and A. van Harten in [EvH], chapter 7.

We will now give a short description of the way in which the main results were obtained.

As mentioned earlier, the KdV has soliton solutions. Moreover, for large classes of initial functions the solutions of the KdV have a soliton character. By this we mean the following:

The solution u(x,t) of (1.1) can be decomposed:

(1.5) $u(x,t) = u_s(x,t) + u_c(x,t)$, *with*

1°) $u_s(x,t)$ *for* $t \to \infty$ *separates into solitary waves of type* (1.2).

$$\sup_{x\in\mathbb{R}} \left| u_s(x,t) + \sum_{n=1}^{N} 2a_n^2 \operatorname{sech}^2 a_n(x - 4a_n^2 t - x_{on}) \right| = o(1) \ , \quad t \to \infty \ .$$

In the above expression given by Tanaka in [T 1], the quantities N and a_n depend only on the initial function U(x).

2°) $u_c(x,t)$ *vanishes on half-lines for* $t \to \infty$.

$$\lim_{\substack{t\to\infty \\ \bar{x}\geq M}} \bar{u}_c(\bar{x},t) = 0 \ , \quad \textit{with}$$

$\bar{x} = x - vt$, $v > 0$ *arbitrary*; $\bar{u}_c(\bar{x},t) = u_c(x,t)$; M *an arbitrary constant.*

This result is also due to Tanaka, see [T 3]. It was obtained in a more rigorous way by Eckhaus and Schuur, [ES]. Moreover, in [S], Schuur improved on the result by showing that $u_c(x,y)$ even vanishes on the half-line $x \geqq -t^{1/3}$.

We show that the solutions of (1.3) display a similar behaviour. Instead of $t \to \infty$ asymptotics, however, we perform $\varepsilon \downarrow 0$ asymptotics on compacta on $1/\delta(\varepsilon)$-timescales with $\delta(\varepsilon) = o(1)$ and $\varepsilon\delta^{-1}(\varepsilon) = O(1)$.

Our theory is built on three basic steps.
In the first step, we determine the structure of u_s and show that u_s separates into solitary waves.
The second step consists of showing that $u - u_s$ vanishes asymptotically.
Finally, we use the results from the first two steps in order to give asymptotical approximations of the solitary waves.

This leads to rigorous results on ε^{-p} timescales, with:
$0 \leqq p < 1$ for solutions containing only one solitary wave, and
$0 \leqq p < \frac{1}{3}$ for solutions containing more than one solitary wave.
We pay special attention to the $\frac{1}{\varepsilon}$-timescale. On this time-scale, we present a consistent theory. Here, by consistency, we mean that in the second step certain conditions on quantities associated with u, are shown to be satisfied for corresponding quantities associated with u_s.

References to a more mathematical description of these results are given in the next section where we give a summary of the contents of this tract.

3. Summary of the contents

II.1:

- Explanation of the IST, including the Lax-approach and the AKNS system.
- Derivation of the S-integrable evolution equations (2.1.14) and (2.1.17). (See Appendix A.1.)

II.2:

- In this section, we will discuss everything we need to know about scattering theory of the *one-dimensional, time independent Schrödinger equation* (S.E.):

$$\left[-\frac{d^2}{dx^2} + u(x)\right]\psi(x) = \lambda\psi(x) \ .$$

  Most of the theory presented here can be found in Eckhaus and van Harten, [EvH].
  Much attention is paid to the asymptotic behaviour of the spectral data. A survey is given of mutual relations between the spectral data, respectively, of relations between the spectral data and the potential. Various, possibly known, results are proved in a way that links up with the theory presented in [EvH]. Moreover, some existing results are extended or stated more precisely. We mention: Theorem (2.2.3), (2.2.36) and Theorems (2.2.4,6,9).

II.3:

- We give the Gel'fand-Levitan or Marchenko equation. This is a linear integral equation by means of which a potential in the S.E. can be recovered from a given set of spectral data.
- The IST is applied to the KdV.
- The 'emergence of solitons' phenomenon is explained.

Chapter III:

- We give evolution equations for the spectral data of a potential u(x,t) that solves the pKdV.
- The decomposition $u(x,t) = u_s(x,t) + u_c(x,t)$ is introduced. We emphasize that this decomposition is based on properties of the S.E., and is not a specific feature of the (p)KdV. What is a specific feature of the (p)KdV is the emergence of solitons from $u_s(x,t)$. This is treated in § 2. Theorem (3.2.1), the theorem that expresses this emergence of solitons,

describes the asymptotic behaviour of $u_s(x,t)$ and x-derivatives of $u_s(x,t)$. The results given in Theorem (3.2.1) are new, also for the KdV itself.

Chapter IV:

- Theorems are given that provide bounds on $u - u_s$ and $\psi_n - \psi_{ns}$ (ψ_n and ψ_{ns}, respectively, are L_2-eigenfunctions of the S.E. with u, respectively u_s, as potential).

  The theorems in § 1 are based on the work of Eckhaus and Schuur, [ES], [S]. In § 2, we take the Trace-formula (2.2.55) as a starting point.

Chapter V:

- This chapter is dedicated to applying the theorems of Chapter IV to the pKdV.

  In § 1 it is shown how we can get results on $\delta^{-1}(\varepsilon)$-timescales by using Theorem (4.1.3), with:

  $\delta(\varepsilon) = \varepsilon^p$, $0 \leq p < 1$ if the solution contains only one soliton,

  $\delta(\varepsilon) = \varepsilon^p$, $0 \leq p < \frac{1}{3}$ if the solution contains one or more solitons.

  These results are given by (5.1.37).

  In § 2, we start by giving a more detailed description of the way by which our main results are obtained. A survey is given of what steps are done and what steps are still to be carried out. Then, we show that we can get estimates for $u - u_s$ on the ε^{-1}-timescale, that are consistent with the conditions of the theorems used. These results are expressed by (5.2.28, 29). They hold for perturbations of type (5.2.6).

Chapter VI:

- In this chapter, we carry out the last step of our perturbation scheme, namely giving approximations for the solitons. For this, we take Theorem (3.2.1) and the consistency results (5.2.28,29) as the starting point. The main result of this chapter is expressed by Theorem (6.1.1).

  On pages 111 - 114, we give a review of the most important results, and the conditions under which they hold, that are needed to obtain the main result of this research. This main result is given by (6.23,24,25). Finally, we conclude the chapter by giving a physical interpretation of the results.

VII.1:

- We discuss the rather trivial perturbation $f(u) = u_{xxx}$. This example is important because it illustrates a possible way of obtaining better approximations of the solitons.

VII.2:

- Here, we show that in the case of a polynomial perturbation (f(u) as in (7.2.1)), many of the calculations needed to obtain the soliton approximations are extremely simple. As examples, we take $f(u) = u$ and $f(u) = \pm u_{xx}$.

VII.3:

- We apply our perturbation scheme to the perturbation (1.4) and find that in this case the solitons of the KdV are good approximations of the solitons of (1.3) on the ε^{-1}-timescale. Moreover, we try to get a solution by substituting a power series in ε. We show that this method of finding solutions is not suitable for solving the pKdV-initial value problem, but can be useful when used in combination with the perturbation scheme.

VII.4:

- We consider the pKdV with $f(u) = u + \frac{1}{2}xu_x$. This pKdV is S-integrable (see § II.1).

  By calculating the pure two-soliton solution, we show how our perturbation scheme can be adapted (in a formal way) to give results that match those obtained by direct integration. (The perturbation scheme must be adapted because $u + \frac{1}{2}xu_x$ does not fall into the class of admissible perturbations (5.2.6)).

CHAPTER II
FUNDAMENTALS

II.1. The Inverse Scattering Transform

For various classes of evolution equations, the Cauchy problem can be solved by means of the Inverse Scattering Transform (IST). We will explain here the essential principle behind the method. To keep the reasoning transparent, we do not bother about technical details.

In their study of the KdV-initial value problem, Gardner, Green, Kruskal and Miura, ([GGKM 1,2]), coupled the KdV equation to the one-dimensional time independent Schrödinger equation (S.E.). They made the S.E. dependent of the time parameter t by taking as a potential the, yet unknown, solution of the KdV.

$$(2.1.1) \quad \begin{cases} u_t - 6uu_x + u_{xxx} = 0 \\ u(x,0) = U(x) \ , \end{cases}$$

$$(2.1.2) \quad \left[-\frac{d^2}{dx^2} + u(x,y)\right]\psi(x,t) = \lambda\psi(x,t) \ .$$

Of course, with the eigenvalue problem

$$(2.1.3) \quad \left[-\frac{d^2}{dx^2} + u(x)\right]v(x) = \lambda v(x) \ ,$$

we can associate a set of spectral data. The problem (2.1.3) has the important property that it admits *inverse-scattering*. That is: Given 'the spectrum S', it is possible to determine the potential u(x) that generates this spectrum.

What must be understood by the spectrum S so that there is a 1-1 relationship between u and S, is explained in the next two sections. Moreover, it will be shown how u(x) can be recovered from S. We already mention that the spectrum partly consists of the set of *eigenvalues*. That is, the set of λ's for which (2.1.3) has a solution $v(x) \in L_2(\mathbb{R})$.

GGKM established the miraculous fact, that in (2.1.2) it is possible to determine how the spectrum depends on the parameter t, without explicit knowledge of the potential u(x,t). In particular, it turned out that the set of eigenvalues is time-independent.

We will illustrate the IST by applying it to the KdV-equation. Consider:

$$\begin{cases} u_t - 6uu_x + u_{xxx} = 0 \\ u(x,0) = U(x) \ . \end{cases}$$

Step 1: Determine the spectrum of U(x).

Step 2: Now, we use the fact that it is possible to determine the evolution of the spectrum of the potential in the S.E., without knowing this potential explicitly. All that is used is that u(x,t) solves the KdV. With initial conditions given by Step 1, this enables us to give the spectrum S(t) at any time t.

Step 2: Since (2.1.3) allows inverse scattering we can determine the potential u(x,t) belonging to the spectrum S(t) at any time t. Because of the 1-1 relationship between u and S the so found potential u(x,t) solves (2.1.1).

A generalization of the above method was given by Lax,([Lax]).
Consider two operators L, M associated with respectively an eigenvalue problem and a time evolution problem:

(2.1.4) a) $Lv = \lambda v$,

b) $v_t = Mv$.

L, M and v depend on the real variable x and the real parameter t. Following GGKM, the eigenvalue parameter λ is taken to be time independent.

(2.1.5) $\lambda_t = 0$.

This implies:

(2.1.6) $L_t v + Lv_t = \lambda v_t$.

Substitution of (2.1.4b) into (2.1.6) leads to a necessary condition for making (2.1.4a) and (2.1.4b) compatible.

(2.1.7) $L_t + LM - ML = 0$.

For a suitable choice of L and M, (2.1.7) represents a (non)-linear evolution equation.
For instance: For $L = -\frac{d^2}{dx^2} + u$, Lax found a hierarchy of possible M's, among which:

$$\text{a)}\quad M = -\frac{d}{dx} \Rightarrow u_t = -u_x \ ,$$

$$\text{b)}\quad M = -4\frac{d^3}{dx^3} + 3u\frac{d}{dx} + \frac{d}{dx}3u \Rightarrow u_t - 6uu_x + u_{xxx} = 0 \qquad \text{(KdV)} \ .$$

Other S-integrable evolution equations can be found by taking other choices of L and M. Of course, L must be so that (2.1.4a) allows inverse scattering.

We conclude this section by mentioning the famous Ablowitz, Kaup, Newell and Segur-system, [AKNS]. AKNS studied the *generalized Zacharov-Shabat eigenvalue problem*, related to a time evolution equation.

$$(2.1.8)\quad \text{a)}\quad v_x = \begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix} v \ , \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \ ,$$

$$\text{b)}\quad v_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} v \ .$$

q and r are potentials depending on the real variable x and the real parameter t. A, B and C are scalar functions of x and t. Again, the eigenvalue parameter ζ is taken time independent:

$$(2.1.9)\quad \zeta_t = 0 \ .$$

The eigenvalue problem (2.1.8a) admits inverse scattering.
The compability conditions are here given by:

$$(2.1.10)\quad \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x} v_i\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} v_i\right) \ , \quad i = 1,2 \ .$$

These conditions can be translated into conditions for A, B and C. Working out these conditions in general leads to yet another condition. This condition is the evolution equation.
Examples of well-known evolution equations that can be solved with the AKNS-system are:

(2.1.11) a) $r = -1$, $q_t + 6qq_x + q_{xxx} = 0$, KdV

b) $r = \pm q$, $q_t \mp 6q^2q_x + q_{xxx} = 0$, modified KdV

c) $r = \pm\bar{q}$, $iq_t = q_{xx} \mp 2q^2\bar{q}$, non-linear Schrödinger

d) $q = -r = -\frac{1}{2}u_x$, $u_{xt} = \sin u$, sine-Gordon

e) $q = r = \frac{1}{2}u_x$, $u_{xt} = \sinh u$, sinh-Gordon

We see that, for $r = -1$, we again find the KdV-equation. This is not remarkable since, for $r = -1$, the system (2.1.8) is equivalent to ($v_2 = \psi$, $q = u$, $\zeta^2 = \lambda$):

(2.1.12) a) $\psi_{xx} + (\lambda + u)\psi = 0$,

b) $\psi_t = A\psi + B\psi_x$.

So far we have dealt with the eigenvalue parameters λ, respectively ζ, as being time-independent. However, this is not an essential condition for being able to apply the IST. In Appendix A.1, we work with the system (2.1.12) and take:

(2.1.13) $\lambda_t = f(\lambda)$.

For instance, for $\lambda_t = 0$ we find:

(2.1.14) $u_t + 6uu_x + u_{xxx} = \beta\{u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x\}$.

We also find the following class of evolution equations:

(2.1.15) $$u_t = \frac{1}{2}\frac{\partial^3}{\partial x^3}(N^{(\lambda)}u) + \frac{u}{\lambda}f(\lambda) + 2(\lambda + u)\frac{\partial}{\partial x}(N^{(\lambda)}u) + \\ + (b_0^{(\lambda)}(t) + \frac{x}{2x}f(\lambda) + N^{(\lambda)}u)u_x .$$

Here the function $b_0^{(\lambda)}(t)$ and the operator $N^{(\lambda)}$ are free to choose under the following restrictions:

(2.1.16) a) The right-hand side in (2.1.15) is λ-independent,

b) $\lim_{x\to\infty} (N^{(\lambda)}u)(x) = 0$.

Examples are:

(2.1.17) a) $\lambda_t = p\lambda$, $b_0^{(\lambda)} = C\lambda$, $N^{(\lambda)}u = -\frac{1}{2}Cu \Rightarrow$

$$u_t + \frac{3}{2} Cuu_x + \frac{1}{4} Cu_{xxx} = p(u + \tfrac{1}{2}xu_x) \ ,$$

b) $$\lambda_t = 2p\lambda^2 \ , \quad b_0^{(\lambda)} = C\lambda \ , \quad N^{(\lambda)}u = \tfrac{1}{2}p\left(\int_x^\infty u(y,t)dy - xu \right) - \tfrac{1}{2}Cu \Rightarrow$$

$$u_t + \frac{3}{2}Cuu_x + \frac{1}{4}Cu_{xxx} = p\left\{ \frac{1}{2}u_x \int_x^\infty u\,dy - u_{xx} - \frac{1}{4}xu_{xxx} - 2u^2 - \frac{3}{2}xuu_x \right\} .$$

The reason why we explicitly mention equations (2.1.14,17) is that for small β or p they represent perturbations on the KdV-equations. The equations (2.1.17) are also given in [CD 4], but there they are derived in a different way from that presented here.
In Appendix A.1 we also determine the evolution of the spectrum of potentials u(x,t) satisfying (2.1.14), respectively (2.1.15,17). Therefore, it is advisable to read § II.2 before studying Appendix A.1. In that section, we will treat the scattering properties of the S.E. in detail.

II.2. Scattering properties of the one-dimensional, time-independent Schrödinger Equation

In this section, we present all the properties of the 'spectrum' of the S.E. that will be needed later on. Of all the properties and theorems given here without proof or reference, a proof can be found in [EvH], chapter 4.

The one-dimensional time-independent Schrödinger equation is given by:

(2.2.1) $$\psi'' + (\lambda - u)\psi = 0 \ , \quad ' = \frac{d}{dx} \ , \quad x \in \mathbb{R} \ .$$

u is a real function, called the potential, and λ is a spectral parameter. We consider potentials that satisfy the following conditions:

(2.2.2) a) $u \in C(\mathbb{R})$,

b) $$\lim_{|x|\to\infty} |u(x)| = 0 \ ,$$

c) $$\int_{-\infty}^{\infty} |u(x)|\,(1 + |x|)^m\,dx < \infty \ .$$

The last condition is called a *growth condition on* u *of order* m, and if u satisfies such a condition, we note this by u = [m].

We write:

(2.2.3) $\lambda = k^2$, with $k \in \bar{\mathbb{C}}_+$ (meaning $\operatorname{Im} k \geqq 0$) .

In the rest of the study we will always take $k \in \bar{\mathbb{C}}_+$, unless stated otherwise. We define solutions ψ_r and ψ_ℓ of (2.2.1) by:

(2.2.4) $\psi_r(x,k) = R(x,k)e^{-ikx}$,

where $R(x,k)$ satisfies:

(2.2.5) a) $R'' - 2ikR' = uR$,

b) $\lim_{x\to-\infty} R(x,k) = 1$, $\lim_{x\to-\infty} R'(x,k) = 0$

and

(2.2.6) $\psi_\ell(x,k) = L(x,k)e^{ikx}$

where $L(x,k)$ satisfies:

(2.2.7) a) $L'' + 2ikL' = uL$,

b) $\lim_{x\to\infty} L(x,k) = 1$, $\lim_{x\to\infty} L'(x,k) = 0$.

In the following we will restrict ourselves to properties of $R(x,k)$. Analogous results hold for $L(x,k)$.

We have the following important theorem.

<u>Theorem (2.2.1):</u>
If $u = [0]$, *then for* $k \in \bar{\mathbb{C}}_+ \setminus \{0\}$ *the problem for* R *has a unique solution in the space of continuous functions of* x, *which are bounded for* $x \to -\infty$. *This solution satisfies* (2.2.5) *in classical sense. Moreover:*

a) R, R', R'' *are continuous in* (x,k) *on* $\mathbb{R} \times \bar{\mathbb{C}}^+ \setminus \{0\}$ *and analytic in* k *on* $\mathbb{C}_+$ (i.e. $\operatorname{Im} k > 0$) *for each* $x \in \mathbb{R}$.

b) *If* $u = [1]$ *then the theorem also holds for* $k = 0$ *and* R, R', R'' *are continuous on* $\mathbb{R} \times \bar{\mathbb{C}}_+$.

c) *If* $u = [2]$, *then also* R_k, R_k', R_k'' *are continuous in* (x,k) *on* $\mathbb{R} \times \bar{\mathbb{C}}_+$. ($R_k = \frac{\partial}{\partial k} R$.)

Since the solution of (2.2.5) is unique, we get:

$$\bar{R}(x,k) = R(x,-\bar{k}) \ ; \ \bar{\psi}_r(x,k) = \psi_r(x,-\bar{k}) \ . \tag{2.2.8}$$

So R and ψ_r are real for k on the positive imaginary axis.

Completely analogous results can be given for L(x,k).
Another important remark is that for $k \in \mathbb{R}\setminus\{0\}$ the functions ψ_r and $\bar{\psi}_r$ are two linearly independent solutions of (2.2.1). The same is true for ψ_ℓ and $\bar{\psi}_\ell$.
So, because (2.2.1) is a second order ODE, we can define the functions ℓ_+, ℓ_-, r_+, r_- of $k \in \mathbb{R}\setminus\{0\}$ by:

$$\begin{aligned} \psi_\ell &= \ell_-\psi_r + \ell_+\bar{\psi}_r \ , \\ \psi_r &= r_+\psi_\ell + r_-\bar{\psi}_\ell \ . \end{aligned} \tag{2.2.9}$$

We can describe the asymptotic behaviour for $|x| \to \infty$ of ψ_r, ψ_r', ψ_ℓ, ψ_ℓ' with $k \in \mathbb{R}\setminus\{0\}$ fixed:

$$\begin{aligned} \psi_\ell(x,k) &\sim e^{ikx} && \text{for } x \to \infty \\ &\sim \ell_+(k)e^{ikx} + \ell_-(k)e^{-ikx} && \text{for } x \to -\infty \\[2ex] \frac{1}{ik}\,\psi_\ell'(x,k) &\sim e^{ikx} && \text{for } x \to \infty \\ &\sim \ell_+(k)e^{ikx} - \ell_-(k)e^{-ikx} && \text{for } x \to -\infty \\[2ex] \psi_r(x,k) &\sim e^{-ikx} && \text{for } x \to -\infty \\ &\sim r_+(k)e^{ikx} + r_-(k)e^{-ikx} && \text{for } x \to \infty \\[2ex] \frac{1}{ik}\,\psi_r'(x,k) &\sim -e^{ikx} && \text{for } x \to -\infty \\ &\sim r_+(k)e^{ikx} - r_-(k)e^{-ikx} && \text{for } x \to \infty \ . \end{aligned} \tag{2.2.10}$$

Using (2.2.10) and the fact that the *Wronskian* of two linearly independent solutions ψ_1, ψ_2 of the S.E.

$$W(\psi_1,\psi_2) = \psi_1\psi_2' - \psi_1'\psi_2 \ , \tag{2.2.11}$$

is a constant for $x \in \mathbb{R}$, it is easily proved that:

(2.2.12) $r_-(k) = \ell_+(k) = \frac{1}{2ik} W(\psi_r,\psi_\ell) = \frac{1}{2ik}\{RL' - L'R + 2ikRL\}$, $k \in \mathbb{R}\setminus\{0\}$,

(2.2.13) $r_+(k) = -\bar{\ell}_-(k) = \frac{1}{2ik} W(\bar{\psi}_\ell,\psi_r) = \frac{1}{2ik}\{\bar{L}R' - R\bar{L}'\}\, e^{-2ikx}$, $k \in \mathbb{R}\setminus\{0\}$.

Since we will make use of the above Wronskians later on, we define:

(2.2.14) $W(k) = W(\psi_r,\psi_\ell)$; $\hat{W}(k) = W(\bar{\psi}_\ell,\psi_r)$.

It is obvious that we can extend the range of definition of $r_-(k)$ to $k \in \bar{\mathbb{C}}_+\setminus\{0\}$.

Moreover, we have the following, very useful, integral expressions for $r_-(k)$ and $r_+(k)$:

(2.2.15) $r_-(k) = 1 - \frac{1}{2ik}\int_{-\infty}^{\infty} u(y)R(y,k)dy$, $k \in \bar{\mathbb{C}}^+\setminus\{0\}$,

(2.2.16) $r_+(k) = \frac{1}{2ik}\int_{-\infty}^{\infty} e^{-2iky}\, u(y)R(y,k)dy$, $k \in \mathbb{R}\setminus\{0\}$.

We now define the following quantities:

(2.2.17) *For* $k \in \bar{\mathbb{C}}_+\setminus\{0\}$, *with* $r_-(k) \neq 0$: $a(k) := \frac{1}{r_-(k)}$.
$a(k)$ *is called the (right) transmission coefficient.*

(2.2.18) *For* $k \in \mathbb{R}\setminus\{0\}$: $b(k) := \frac{r_+(k)}{r_-(k)}$.
$b(k)$ *is called the (right) reflection coefficient.*

(2.2.19) *For* $k \in \mathbb{R}\setminus\{0\}$: $\psi(x,k) := a(k)\psi_r(x,k)$ *is the solution of the S.E. with asymptotic behaviour:*

(2.2.20)
$$\psi(x,k) \sim ae^{-ikx} \quad \textit{for } x \to -\infty$$
$$\sim e^{-ikx} + be^{ikx} \quad \textit{for } x \to \infty .$$

These definitions are motivated by the fact that for k real and positive, we have the following physical interpretation:
$\psi e^{-i\lambda t}$ represents a wave coming from the right of which an amplitude fraction $|a(k)|$ travels towards $-\infty$ and an amplitude fraction $|b(k)|$ is scattered back.

Potentials with $b \equiv 0$ play an important role in our analysis. They are called *reflectionless potentials*.
It is easily seen that as an analogon of (2.2.8) we have:

$$(2.2.21) \quad a) \ \bar{r}_-(k) = r_-(-\bar{k}) \ ; \ \bar{a}(k) = a(-\bar{k}) \ , \quad k \in \bar{\mathbb{C}}_+ \setminus \{0\} \ ;$$

$$b) \ \bar{r}_+(k) = r_+(-k) \ ; \ \bar{b}(k) = b(-k) \ , \quad k \in \mathbb{R} \setminus \{0\} \ .$$

Moreover, in accordance with the physical interpretation, we have:

$$(2.2.22) \quad |r_-(k)|^2 = 1 + |r_+(k)|^2 \ , \quad k \in \mathbb{R} \setminus \{0\} \ ,$$

$$|a(k)|^2 + |b(k)|^2 = 1 \ , \quad k \in \mathbb{R} \setminus \{0\} \ .$$

We will now give a review of properties of $R(x,k)$, $r_-(k)$, $a(k)$ and $b(k)$. When a stronger growth condition than $u = [0]$ is needed, this is mentioned.

The problem (2.2.5) for R can be reformulated to an integral equation

$$(2.2.23) \quad R(x,k) = 1 + \int_{-\infty}^{x} G(x,y,k)R(y,k)dy \ ,$$

with

$$(2.2.24) \quad a) \ G(x,y,k) = \frac{u(y)}{2ik}\left(e^{2ik(x-y)} - 1\right) \ , \quad k \in \bar{\mathbb{C}}^+ \setminus \{0\} \ ,$$

$$b) \ G(x,y,0) = u(y)(x-y) \ ; \quad u = [1] \ .$$

$R(x,k)$ can be given as a Neumann series

$$(2.2.25) \quad R(x,k) = \sum_{n=0}^{\infty} G_n(x,k) \ , \quad k \in \bar{\mathbb{C}}_+ \setminus \{0\} \ ,$$

with

$$(2.2.26) \quad G_0 = 1 \ , \quad G_{n+1}(x,k) = \int_{-\infty}^{x} G(x,y,k)G_n(y,k)dy \ .$$

The functions G_n satisfy the following bound:

$$(2.2.27) \quad |G_n(x,k)| \leq \frac{U_0(x)}{n! \ |k|^n} \leq \frac{U_0}{n! \ |k|^n} \ ,$$

with

$$(2.2.28)\quad U_0(x) = \int_{-\infty}^{x} |u(y)|\,dy \;; \quad U_0 = \int_{-\infty}^{\infty} |u(y)|\,dy \;.$$

Also R'(x,k) can be presented as a Neumann series:

$$(2.2.29)\quad R'(x,k) = \sum_{n=0}^{\infty} \hat{G}_n(x,k) \;, \quad k \in \bar{\mathbb{C}}_+ \;,$$

with

$$\hat{G}_n(x,k) = \int_{-\infty}^{x} G'(x,y,k)G_n(y,k)\,dy \;, \quad ' = \frac{\partial}{\partial x} \;.$$

The following theorem holds.

Theorem (2.2.2):

The series (2.2.25) *and* (2.2.29) *represent convergent asymptotic expansions for* $|k| \to \infty$. *The* n-*th term is of order* $|k|^{-n}$ *and the first* N *terms approximate* R(x,k), *respectively* R'(x,k), *with order* $|k|^{-N-1}$ *uniformly in* x *on* $\mathbb{R}$.

A trivial consequence of (2.2.25,27) is that:

$$(2.2.30)\quad |R(x,k)| \leq e^{U_0(x)/|k|} \;, \quad k \in \bar{\mathbb{C}}_+ \setminus \{0\} \;.$$

Some other properties of R(x,k) are summarized in a theorem.

Theorem (2.2.3):

If $u = [0]$; $u \in C^m(\mathbb{R})$; $u^{(p)}(x)$ *is bounded for* $x \to -\infty$, $0 \leq p \leq m$, *then:*

$$(2.2.31)\quad R^{(p+1)}(x,k) = \int_{-\infty}^{x} e^{2ik(x-y)} \frac{\partial^p}{\partial y^p}(u(y)R(y,k))\,dy \;, \quad k \in \bar{\mathbb{C}}^+ \setminus \{0\} \;, \quad 0 \leq p \leq m+1 \;.$$

If moreover $u^{(p)} = [0]$, $0 \leq p \leq m$, *then:*

$$(2.2.32)\quad \text{a)}\quad |R^{(p)}(x,k)| \leq Ce^{U_0/|k|} \;, \quad k \in \bar{\mathbb{C}}^+ \setminus \{0\} \;, \quad 0 \leq p \leq m+1 \;,$$

$$\text{b)}\quad R^{(p)}(x,k) = O\left(\frac{1}{|k|}\right) \;, \quad |k| \to \infty \;, \quad k \in \bar{\mathbb{C}}^+ \;, \quad 1 \leq p \leq m \;,$$

uniformly in x *on* $\mathbb{R}$.

Proof:

Given in Appendix A.2.

We also have bounds for R that are uniformly valid in k:

(2.2.33) a) $|R(x,k)| \leqq B(1+x_+)$, $u = [1]$, $(x,k) \in \mathbb{R} \times \bar{\mathbb{C}}_+$;

b) $|R'(x,k)| \leqq B$, $u = [1]$, $(x,k) \in \mathbb{R} \times \bar{\mathbb{C}}_+$.

B is a constant depending only on u(x). $x_+ = \max\{0,x\}$.

Finally, we specify the asymptotic behaviour of R(x,k) for $x \to \infty$. We have:

(2.2.34) $\lim_{x\to\infty} R(x,k) = r_-(k)$; $\lim_{x\to\infty} R'(x,k) = 0$,

both limits are uniform in k on compacta $\subset \mathbb{C}_+$.

We will now focus our attention on $r_-(k)$, a(k) and b(k).
As a corollary to Theorem (2.2.1), $r_-(k)$ and a(k) have the following smoothness properties:

(2.2.35) $r_-(k)$ *is analytic on* $\mathbb{C}_+$,

$r_-(k)$ *is continuous on* $\bar{\mathbb{C}}_+ \setminus \{0\}$,

a(k) *is meromorphic on* $\mathbb{C}_+$ *with poles at the zeros of* $r_-(k)$,

a(k) *is continuous on* $\bar{\mathbb{C}}_+ \setminus \{0, \text{zeros of } r_-(k)\}$.

If u = [2], then a(k) is also continuous in k = 0.
To be precise, we have:

(2.2.36) a) *If* $W(0) \neq 0$, *then*

$$a(k) \sim \frac{2ik}{W(0)} \quad \text{for } k \to 0,\ k \in \bar{\mathbb{C}}_+ ; \quad b(0) = -1 ,$$

b) *If* $W(0) = 0$, *then* $|\frac{dW}{dk}(0)| \geqq 2$ *and*

$$a(0) = 2i\left(\frac{dW}{dk}(0)\right)^{-1} ; \quad b(0) = \frac{d\tilde{W}(0)}{dk}\left(\frac{dW}{dk}(0)\right)^{-1} .$$

The case $W(0) \neq 0$ (i.e. $\psi_r(x,0)$ and $\psi_\ell(x,0)$ are linearly independent) is referred to as the *generic case*, while $W(0) = 0$ is referred to as the *exceptional case*.
Proof of (2.2.36) is given in Appendix A.2. (The proof is analogous to that of 'Corollary of theorem 4.2.5.I' in [EvH].)

For the asymptotic properties of $r_-(k)$, $a(k)$ and $b(k)$ we have the following results:

$$(2.2.37)\quad a)\quad r_-(k) = 1 - \frac{1}{2ik}\int_{-\infty}^{\infty} u(y)\,dy + O\left(\frac{1}{|k|^2}\right), \quad |k| \to \infty,\ k \in \bar{\mathbb{C}}_+ \setminus \{0\},$$

$$b)\quad a(k) = 1 + \frac{1}{2ik}\int_{-\infty}^{\infty} u(y)\,dy + O\left(\frac{1}{|k|^2}\right), \quad |k| \to \infty,\ k \in \bar{\mathbb{C}}_+ \setminus \{0\}.$$

(2.2.37) is a trivial corollary of (2.2.15,25) and Theorem (2.2.2).

Some other asymptotical results are given in the next theorem.

Theorem (2.2.4):

a) *If* $u \in C^1(\mathbb{R})$ *and* $u^{(p)}(x) = [0]$, $p = 0,1$, *then*

$$i)\quad |r_-(k)|^2 = 1 + O\left(\frac{1}{|k|^2}\right), \quad |k| \to \infty,\ k \in \bar{\mathbb{C}}_+,\ \textit{if} \int_{-\infty}^{\infty} u(x)\,dx \neq 0,$$

$$ii)\quad |r_-(k)|^2 = 1 + O\left(\frac{1}{|k|^4}\right), \quad |k| \to \infty,\ k \in \bar{\mathbb{C}}_+,\ \textit{if} \int_{-\infty}^{\infty} u(x)\,dx = 0.$$

b) *If* $u \in C^m(\mathbb{R})$; $u^{(p)}(x) = [0]$, $0 \leq p \leq m$; $u^{(p)}(x)$ *is bounded for* $x \to -\infty$, $0 \leq p \leq m$, *then*

$$|b(k)|^2 = O(|k|^{-2(m+1)}), \quad |k| \to \infty,\ k \in \mathbb{R},$$

$$|a(k)| = 1 + O(|k|^{-2(m+1)}), \quad |k| \to \infty,\ k \in \mathbb{R}.$$

Proof:

See Appendix A.2.

Finally, we give conditions under which $b(k)$ can be extended to a meromorphic function on a strip in the complex k-plane.
With (2.2.15,16,30,33) and Theorem (2.2.1) it follows that:

(2.2.38) *If* $u = [1]$ *and* $\lim_{x\to\infty} u(x)e^{2\mu x} < \infty$, *then*

i) $b(k)$ *is meromorphic on* $0 < \text{Im } k < \mu$, *with poles in the zeros of* $r_-(k)$,

ii) $b(k)$ *is continuous on* $\{k \in \bar{\mathbb{C}}_+ \setminus \{\text{zeros of } r_-(k)\} \mid \text{Im } k \leq \mu\}$,

iii) $b(k) = O(|k|^{-1})$, $|k| \to \infty$, $0 \leq \text{Im } k < \min\{\mu, k_1\}$,
where k_1 *is such that* $r_-(k) \neq 0$ *for* $0 \leq \text{Im } k < k_1$.

From [CD 4], § 2.1, we know that:

(2.2.39) *If* $\lim_{x\to\pm\infty} u(x)e^{\pm 2\mu_\pm x} = 0$, *then*, $b(k)$ *is meromorphic in the so-called 'Bargmann strip'*; $-\min\{\mu_-, \mu_+\} < \text{Im } k < \mu_+$, *with poles at the zeros of* $r_-(k)$.

We now turn our attention to the so-called *discrete spectrum* of the S.E. That is, we look for values of λ for which the S.E. has a solution in $L_2(\mathbb{R})$. These λ's are called *eigenvalues*. The corresponding L_2-solutions of the S.E. are called *eigenfunctions*. (Solutions of the S.E. that are not in L_2 are often called *generalized eigenfunctions*.)
We have the following important theorem:

Theorem (2.2.5):
If $u = [1]$ *then*

1°. *The number* N *of eigenvalues is finite.*
2°. *They are given by* $\lambda_n = (ik_n)^2$ *with* $k_n \in \mathbb{R}_+$ *and* $r_-(ik_n) = 0$.
(r_(k) has no other zeros.)
Each eigenvalue is simple. That is: The eigenspace $E(\lambda_n)$ *is one-dimensional and is spanned by the real function* $\psi_r(x, ik_n)$.
So for $k = ik_n$ *there exists* $\alpha(k) \in \mathbb{R} \setminus \{0\}$ *with* $\psi_r(x,k) = \alpha(k)\psi_\ell(x,k)$.
(Terminology: The spectrum of the S.E. is non-degenerate.)

In the following the eigenvalues are ordered by

(2.2.40) $0 < k_1 < k_2 < \ldots < k_N$.

With these eigenvalues we define the following quantities:

(2.2.41) $\tilde{\psi}_n(x) = \psi_r(x, ik_n)$.

(2.2.42) a) $\tilde{c}_n = \lim_{x\to\infty} \tilde{\psi}_n(x)e^{k_n x} = \alpha(ik_n)$,

b) $\gamma_n = \int_{-\infty}^{\infty} \tilde{\psi}_n^2(x)dx$.

(2.2.43) $\psi_n(x) = \gamma_n^{-\frac{1}{2}} \tilde{\psi}_n(x)$.

(2.2.44) a) $c_n = \lim_{x\to\infty} \psi_n(x)e^{k_n x}$,

b) $d_n = \lim_{x\to-\infty} \psi_n(x)e^{-k_n x}$.

Note that

(2.2.45) a) $\int_{-\infty}^{\infty} \psi_n^2(x)dx = 1$,

b) $c_n = \gamma_n^{-\frac{1}{2}} \tilde{c}_n$; $d_n = \gamma_n^{-\frac{1}{2}}$.

In the following we will mostly work with the eigenfunctions $\psi_n(x)$. We therefore define:

(2.2.46) *The normalization coefficient associated with the eigenvalue* $\lambda_n = -k_n^2$, *is the value* c_n *as defined in* (2.2.44).

With ψ_n we can associate a second solution $\phi_n(x)$ of the S.E., which is linearly independent of ψ_n. We choose $\phi_n(x)$ such that the asymptotic behaviour of ϕ_n is specified by:

(2.2.47) a) $\lim_{x\to\infty} \phi_n\ e^{-k_n x} = -\frac{1}{c_n}$,

b) $\lim_{x\to-\infty} \phi_n\ e^{k_n x} = \frac{1}{d_n}$.

Or equivalently

(2.2.48) a) $\lim_{x\to\infty} \phi_n \psi_n = -1$,

b) $\lim_{x\to-\infty} \phi_n \psi_n = 1$.

Of course, the asymptotic behaviour of ϕ_n for $x \to -\infty$ is determined by the asymptotic behaviour of ϕ_n for $x \to \infty$ (and vice versa) by the Wronskian:

$$(2.2.49) \quad W(\psi_n, \phi_n) = -2k_n \; .$$

There are a number of important equalities that exhibit relations between the discrete spectrum $(\lambda_n; \psi_n(x))$, the continuous spectrum $(b(k); \psi(x,k),$ $k \in \mathbb{R})$ and the potential $u(x)$.
We have:

$$(2.2.50) \quad r_-(k) = \exp\left\{\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log |a(q)|^2}{k-q} dq\right\} \prod_{n=1}^{N} \frac{k-ik_n}{k+ik_n} \; , \qquad \operatorname{Im} k > 0 \; ,$$

$$r_-(k) = \lim_{\varepsilon \downarrow 0} r_-(k+i\varepsilon) \; , \qquad k \in \mathbb{R} \; .$$

This expression is given by Zacharov & Faddeev in [ZF]. It is derived from the knowledge of the zeros of $r_-(k)$ and the analyticity of $r_-(k)$ for $k \in \mathbb{C}_+$.
Note: If u = [2], we have no convergency problems in q = 0 because of (2.2.36). Because of Theorem (2.2.4b) (m = 0), we have no convergence problem at infinity.

The next expression follows from (2.2.50) and Lemma (4.3.4) in [EvH], which says that:

$$(2.2.51) \quad \frac{dr_-}{dk} = \frac{1}{i\alpha(k)} \int_{-\infty}^{\infty} \psi_r^2(x,k)\,dx \; , \quad \text{for } k = ik_n \; .$$

We get:

$$(2.2.52) \quad \gamma_n = \delta_n \exp\left\{-\frac{k_n}{\pi} \int_0^{\infty} \frac{\log(1-|b(k)|^2)}{k^2+k_n^2} dk\right\} \; ,$$

with

$$\delta_n = \frac{\tilde{c}_n}{2k_n} \prod_{\substack{p=1 \\ p\neq n}}^{N} \left(\frac{k_n - k_p}{k_n + k_p}\right) .$$

Using (2.2.51), we can reformulate formula (20) on page 20 of [NMPZ], so that in our notation we get:

$$\psi_\ell(x,k) = e^{ikx}\left\{1 - i\sum_{n=1}^{N} \frac{c_n \psi_n(x) e^{-k_n x}}{k+ik_n} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{b(-k')\psi_\ell(x,-k')e^{-ik'x}}{k'-k-io} dk'\right\} .$$

Now, using $\psi(x,k) = b(k)\psi_\ell(x,k) + \bar{\psi}_\ell(x,k)$, this leads to:

$$(2.2.53)\quad \psi(x,k) = b(k)e^{ikx}\left\{1 - i\sum_{n=1}^{N}\frac{c_n\psi_n(x)e^{-k_nx}}{k+ik_n}\right\} + $$

$$-\frac{b(k)e^{ikx}}{2\pi i}\left\{\int_{-\infty}^{\infty}\frac{b(-k')\psi_\ell(x,-k')e^{-ik'x}}{k'-k-io}\,dk'\right\} +$$

$$+ e^{-ikx}\left\{1 + i\sum_{n=1}^{N}\frac{c_n\psi_n(x)e^{-k_nx}}{k-ik_n} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{b(k')\psi_\ell(x,k')e^{ik'x}}{k'-k+io}\,dk'\right\}.$$

For reflectionless potentials this reduces to the more attractive formula

$$(2.2.54)\quad \psi(x,k) = e^{-ikx}\left\{1 - \sum_{n=1}^{N}\frac{c_n\psi_n(x)e^{-k_nx}}{k_n+ik}\right\}.$$

The following equality is very important for our work. It is called the *Trace formula*:

$$(2.2.55)\quad u(x) = -4\sum_{n=1}^{N}k_n\psi_n^2(x) - \frac{2i}{\pi}\int_{-\infty}^{\infty}kb(-k)\,\frac{\psi^2(x,k)}{|a(k)|^2}\,dk\,.$$

This formula is derived by Deift & Trubowitz, [DT].
Sufficient conditions for the formula to hold are: $u \in C^2(\mathbb{R})$; $u = [1]$; $u^{(p)} = [0]$, $p = 0,1$.

The following theorem provides us with a set of expressions that relate integrals over polynomials in u and x-derivatives of u, with the spectral data of u.

<u>Theorem (2.2.6):</u>
If u *satisfies*

$$u \in C^m(\mathbb{R})\ ;\quad u = [2]\ ;\quad u^{(p)} = [0]\ ,\quad 1 \leq p \leq m\ ;$$

$$\lim_{|x|\to\infty} u^{(p)}(x) = 0\ ,\quad 0 \leq p \leq m\ ,$$

then

$$(2.2.56)\quad \frac{1}{(2i)^{2n+1}}\int_{-\infty}^{\infty}\sigma_{2n+1}\,dx = \frac{1}{\pi i}\int_{0}^{\infty}k^{2n}\log(1-|b(k)|^2)dk +$$

$$- \frac{2}{2n+1} \sum_{\ell=1}^{N} (ik_\ell)^{2n+1} \quad , \qquad 0 \leq n \leq [\tfrac{m}{2}] \ ,$$

where σ_n *is defined as*

$$\sigma_1(x) = - u(x) \ ; \quad \sigma_2(x) = - \frac{du}{dx}(x) \ ;$$

$$\sigma_{n+1} = \frac{d}{dx}\sigma_n + \sum_{i+j=n} \sigma_i\sigma_j \ , \qquad n \geq 2 \ .$$

Remark:
This theorem is based on the work of Zacharov & Faddeev, [ZF]. These authors show that an infinite set of integrals for the KdV-equation is given by (2.2.56) with $m = \infty$. In their presentation, however, it is not emphasized that in fact (2.2.56) is a property of the S.E. (Indeed, if a potential $u(x,t)$ evolves with t according to the KdV-equation, then the eigenvalues as well as $|b(k,t)|$ do not change in time and (2.2.56) represents a set of integrals for the KdV.) Moreover, in [ZF] the analysis is performed under the conditions that: 'u $\in C^\infty(\mathbb{R})$ and along with its derivatives decreases rapidly'. This gives no insight into the number of equalities of type (2.2.56) that hold under less stringent conditions.

Proof of Theorem (2.2.6):
Given in Appendix A.2.

We explicitly mention the equalities obtained from (2.2.56) with n = 0,1. We have:

(2.2.57) a) $$\int_{-\infty}^{\infty} u(x)dx = -4 \sum_{n=1}^{N} k_n - \frac{2}{\pi} \int_0^{\infty} \log(1 - |b(k)|^2)dk \ ,$$

b) $$\int_{-\infty}^{\infty} u^2(x)dx = \frac{16}{3} \sum_{n=1}^{N} k_n^3 - \frac{8}{\pi} \int_0^{\infty} k^2 \log(1 - |b(k)|^2)dk \ .$$

We conclude this summary with a theorem about reflectionless potentials, see [GGKM 2].

Theorem (2.2.7):
Given an arbitrary set of positive numbers c_n, $n = 1,\dots,N$, *and an arbitrary set of mutually different positive numbers* k_n, $n = 1,\dots,N$.
Define the $N \times N$-*matrix* $I + C(x)$ *by*

(2.2.58) I = identity , $C = [c_{mn}]_{m,n=1,\dots,N}$ *with* $c_{mn} = \frac{c_m c_n}{k_m + k_n} e^{-(k_m+k_n)x}$.

Then:

(2.2.59) $u(x) = -2 \frac{d^2}{dx^2} \log \det(I + C(x))$ *is* *the* *reflectionless potential*

with eigenvalues k_n *and normalization coefficients* c_n.

The eigenfunctions $\psi_n(x)$ *are given by*

(2.2.60) $\psi_n(x) = \frac{1}{\det(I + C(x))} \sum_{m=1}^{N} c_m e^{-k_m x} Q_{mn}(x)$,

where Q_{mn} *are the cofactors of* $I + C(x)$.

For future purposes we mention the following lemma.

Lemma (2.2.1):
Let $I + C(x)$ *be defined by* (2.2.28), *then:*
$\det(I + C(x))$ *is a polynomial in* $c_n^2 e^{-2k_n x}$, $n = 1,\dots,N$, *with positive coefficients. The* 0*-th order term equals* 1.

Proof:
The proof is elementary, but it is presented in Appendix A.2 for the sake of completeness.

The final part of this section deals with parameter dependent potentials $u = u(x,t)$.
First, we introduce the following notation.

(2.2.61) $u = [m]_u$ means that u(x,t) satisfies a growth condition of order m in x, uniformly in t on the time regions under consideration.

From (2.2.30) and Theorem (2.2.3) it is seen that:

(2.2.62) *If* $u(\cdot,t) \in C^m(\mathbb{R})$; $u^{(p)} = [0]_u$, $0 \leq p \leq m$; $u^{(p)}$ *is bounded for* $x \to -\infty$, $0 \leq p \leq m$, *then:*

$$|R^{(p)}(x,k,t)| \leq C e^{U_0/|k|} , \quad k \in \bar{\mathbb{C}}_+ \setminus \{0\} , \quad 0 \leq p \leq m+1 ,$$

where U_0 *is so that* $\int_{-\infty}^{\infty} |u(x,t)|dx \leq U_0$, $\forall t$.
(For p = 0 we can take C = 1, for p = 1 we can take $C = U_0$.)

We also have (see (2.2.37)):

(2.2.63) *If* $u = [0]_u$ *then* $a(k,t) = 1 + O(\frac{1}{|k|})$, $|k| \to \infty$, $\mathrm{Im}\, k \geq 0$, *uniformly in* t.

A question that arises when studying parameter dependent potentials is 'where' and 'when' eigenvalues can emerge or can vanish. Concerning this matter we have got the following theorems.
The first theorem can be abstracted from § 4.2.5 and Theorem 4.3.III in [EvH].

Theorem (2.2.8):
If $u(x,t)$ *satisfies*

a) $u \in C(\mathbb{R} \times [T_0,T_1])$, $\max_{t\in[T_0,T_1]} |u(x,t)| \leq \bar{u}(x)$ *and* $\bar{u}(x) = [0]$,

b) $\frac{\partial u}{\partial t} \in C(\mathbb{R} \times [T_0,T_1])$, $\max_{t\in[T_0,T_1]} |\frac{\partial u}{\partial t}(x,t)| \leq \bar{u}_1(x)$ *and* $\bar{u}_1(x) = [0]$,

then for $t \in [T_0,T_1]$ *eigenvalues can only vanish at, or start from,* $k = 0$.
Moreover, if $\bar{u} = [1]$, *then* $\lambda = 0$ *is not an eigenvalue.*

This answers the question 'where' eigenvalues can vanish or emerge.

Theorem (2.2.9):
If $u(x,t) \in C(\mathbb{R} \times [T_0,T_1])$ *with* $\max_{t\in[T_0,T_1]} |u(x,t)| \leq \bar{u}(x)$ *and* $\bar{u}(x) = [1]$,
then:
For $t \in (T_0,T_1)$, *eigenvalues can only emerge or vanish at times* t_0 *for which* $W(0,t_0) = 0$.

Proof:
In a completely analogous way to the proof of Theorem 4.2.5, I(a), given in [EvH], it is proved that:

$\frac{\partial^i}{\partial x^i} R(x,k,t)$ and $\frac{\partial^i}{\partial x^i} L(x,k,t)$ are continuous in (x,k,t) on $\mathbb{R} \times \bar{\mathbb{C}}_+ \times [T_0,T_1]$,

for $i = 0,1,2$.
Now suppose an eigenvalue emerges or vanishes at time $t = 0$, with $W(0,t_0) \neq 0$.

Consider $u(x,t)$ on $\mathbb{R} \times [t_0-\delta, t_0+\delta] \subset \mathbb{R} \times [T_0, T_1]$. Because of the continuity of W there is a neighbourhood $U \subset \bar{\mathbb{C}}_+ \times [t_0-\delta, t_0+\delta]$ of $(0, t_0)$ where $W(k,t) \neq 0$. For eigenvalues ($\neq 0$), we know that $r\_(k,t) = \frac{1}{2ik} W(k,t) = 0$.
So, no eigenvalue trajectories can cross the region U.

Q.E.D.

This answers the question 'when' eigenvalues can vanish or emerge.

II.3. Inverse scattering for the Schrödinger Equation; The IST applied to the KdV-initial value problem

We have seen how we can associate a set of spectral data with a potential in the S.E.

(2.3.1) $$S = \{\{k_n, c_n\}_{n=1,\dots,N};\ b(k),\ k \in \mathbb{R}\} .$$

It is also possible to find the potential u belonging to a given set of spectral data S. This can be done with the *Gel'fand-Levitan equation*, [GL].

(2.3.2) $$\beta(y,x) + \Omega(x+y) + \int_0^\infty \Omega(x+y+z)\beta(z,x)dz = 0 , \quad y > 0 .$$

A similar equation corresponding to a somewhat different inverse scattering context was derived by Marchenko, [M]. In consequence, (2.3.2) also occurs under the name Marchenko equation.
In this linear integral equation, the unknown function β depends on a variable y and a parameter x. Ω is defined as

(2.3.3) $$\Omega(\xi) = \Omega_d(\xi) + \Omega_c(\xi) ;$$

$$\Omega_d(\xi) = 2 \sum_{n=1}^{N} c_n^2 e^{-2k_n\xi} ,$$

$$\Omega_c(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} b(k) e^{2ik\xi}\, dk .$$

The potential u with spectral data S is given by:

(2.3.4) $$u(x) = -\frac{d}{dx} \beta(0^+, x) .$$

As a matter of fact, solving the Gel'fand-Levitan equation is not trivial.

Only for $b \equiv 0$ can explicit solutions be obtained. These solutions are given by (2.2.59).

Proof is given in [EvH], §§ 4.4,4.5, of the following important results.

(2.3.5) *If a potential* u *satisfies* u = [2], *then,* (2.3.2) *has a unique solution* $\beta \in F(\mathbb{R} \to L_2(0,\infty))$. ($F(\mathbb{R} \to V)$ is the set of functions on $\mathbb{R}$ with values in V.)

Moreover,

(2.3.6) a) $\beta \in C_0(\bar{\mathbb{R}}_+^2) \cap C(\mathbb{R} \to L_1(0,\infty)) \cap C(\mathbb{R} \to L_2(0,\infty))$, *with*

$$C_0(\bar{\mathbb{R}}_+^2) = \{w \in C([0,\infty) \times \mathbb{R}) \mid \forall x \in \mathbb{R} \ \lim_{y\to\infty} w(y,x) = 0\} ,$$

b) $\frac{\partial \beta}{\partial y} \in C_0(\bar{\mathbb{R}}_+^2)$; $\frac{\partial \beta}{\partial x} \in C_0(\bar{\mathbb{R}}_+^2)$.

By now, we can specify the steps given in § II.1, needed to solve the KdV-initial value problem (2.1.7) by means of the IST.

Step 1: Determine the set of spectral data S(0) belonging to u(x,0).

$$S(0) = \{\{k_n(0),c_n(0)\}_{n=1,\dots,N};\ b(k,0),\ k \in \mathbb{R}\} .$$

Step 2: From GGKM 1,2 we know that the set of spectral data at time t is given by:

(2.3.7) $S(t) = \{\{k_n(t),c_n(t)\}_{n=1,\dots,N};\ b(k,t),\ k \in \mathbb{R}\}$, with

(2.3.8) $k_n(t) = k_n(0)$; $c_n(t) = c_n(0)e^{4k_n^3 t}$; $b(k,t) = b(k,0)e^{8ik^3 t}$.

Step 3: Use the Gel'fand-Levitan equation to find the potential u(x,t) with spectral data S(t). Because of the 1-1 correspondence between u(x,t) and S(t), this potential is the solution of (2.1.7).

We have seen that, to be able to use the IST, the potential u(x,t) must satisfy certain conditions. In the case of problem (2.1.7) fulfillment of these conditions can be guaranteed by putting conditions on u(x,0).
In [Co], A. Cohen gives the following set of sufficient conditions on u(x,0):

(2.3.9) a) $u(x,0) \in C^3(\mathbb{R})$ *with piecewise continuous fourth derivative.*

b) $u^{(p)}(x,0) = O(|x|^{-M})$, $|x| \to \infty$ *for* $p \leq 4$, *with*
$M > 8$ *in the generic case* $(W(0) \neq 0)$,
$M > 10$ *in the exceptional case* $(W(0) = 0)$.

As was said earlier, the Gel'fand-Levitan equation can be solved explicitly for $b \equiv 0$. Since, from (2.3.8), we know that $b(k,0) = 0$ implies $b(k,t) = 0$, this means that explicit solutions of (2.1.7) can be found by taking $u(x,0)$ reflectionless. These solutions are given by (2.2.59) with k_n and c_n evaluating according to (2.3.8). They are called *N-soliton solutions* on account of their remarkable asymptotic behaviour. The following theorem, given by Tanaka in [T 1], specifies this asymptotic behaviour.

Theorem (2.3.1):
Let $u_s(x,t)$ *and* $\psi_{ns}(x,t)$ *be defined by* (2.2.59), (2.2.60), *with* $k_n(t)$ *and* $c_n(t)$ *given by* (2.3.8). *Then*

$$(2.3.10) \quad \lim_{t\to\pm\infty} \psi_{ns}^2(x,t) - \tfrac{1}{2}k_n \operatorname{sech}^2 k_n(x - 4k_n^2 t - \delta_n^{\pm}) = 0, \text{ uniformly in } x \text{ on } \mathbb{R}.$$

Here $\delta_n^{\pm}$ *are defined by*

$$(2.3.11) \quad \text{a)} \quad \delta_n^{+} = \frac{1}{2k_n} \log \frac{c_n^2(0)}{2k_n} \prod_{i=n+1}^{N} \left(\frac{k_n - k_i}{k_n + k_i}\right)^2 ,$$

$$\text{b)} \quad \delta_n^{-} = \frac{1}{2k_n} \log \frac{c_n^2(0)}{2k_n} \prod_{i=1}^{n-1} \left(\frac{k_n - k_i}{k_n + k_i}\right)^2 .$$

Corollaries:

$$(2.3.12) \quad \text{a)} \quad \lim_{t\to\pm\infty} u_s(x,t) + \sum_{n=1}^{N} 2k_n^2 \operatorname{sech}^2 k_n(x - 4k_n^2 t - \delta_n^{\pm}) = 0 ,$$

uniformly in x *on* $\mathbb{R}$.

$$\text{b)} \quad \lim_{t\to\pm\infty} u_s(x,t) + 2k_n^2 \operatorname{sech}^2 k_n(x - 4k_n^2 t - \delta_n^{\pm}) = 0 ,$$

uniformly in x *on a strip:*

$$(4k_n^2 - \alpha_n)t + \beta \leq x \leq (4k_n^2 + \alpha_{n+1})t + \tilde{\beta} ,$$

with $0 \leq \alpha_n < 4(k_n^2 - k_{n-1}^2)$; $\beta, \tilde{\beta}$ *are constants.*

(2.3.12a) follows from (2.3.10) with (2.2.55).
(2.3.12b) follows from (2.3.12a) and the asymptotic behaviour of $\mathrm{sech}^2 x$.

Remark:

As can be seen by direct substitution, every function of the form $-2a^2 \,\mathrm{sech}^2 a(x-4a^2 t+c)$ satisfies the KdV-equation.

So, the formulas (2.3.12) describe the fact that the solution $u_s(x,t)$ separates into N - so-called - soliton solutions when $t \to \pm\infty$. Moreover, the solitons for $t \to \infty$ only differ from the solitons for $t \to -\infty$ by a change of phase.

This is a truly remarkable phenomenon, since the KdV-equation is non-linear.

The solution $u_s(x,t)$ is called the N-soliton solution.

The phenomenon described by (2.3.12) is referred to as *the emergence of solitons*.

CHAPTER III
EMERGENCE OF SOLITONS FOR THE pKdV

III.1. Evolution of the spectral data for potentials satisfying the pKdV

As pointed out in the introduction, we wish to use the IST in order to study the pKdV-initial value problem:

$$(3.1.1) \qquad \begin{cases} u_t - 6uu_x + u_{xxx} = \varepsilon f(u) \\ u(x,0) = U(x) \ . \end{cases}$$

For an eigenvalue problem, we take the S.E., with potential u(x,t) evolving according to (3.1.1):

$$(3.1.2) \qquad \psi_{xx}(x,t) + [\lambda(t) - u(x,t)]\psi(x,t) = 0 \ .$$

Of course, the existence and uniqueness of a solution of (3.1.1), as well as the possibility of solving (3.1.1) with the IST, will depend on U(x) and f(u).

In this thesis, no attention has been paid to the above problems. We assume f(u) and U(x) to be such that (3.1.1) has a unique solution, that among other conditions satisfies:

(3.1.3) a) u(x,t) *sufficiently smooth.*

b) $u = [0]_u$ *and* $f(u) = [0]_u$ *on the time-regions under consideration.*

c) $u = [2]$ *and* $f(u) = [2]$, $\forall t$.

d) $\lim_{|x|\to\infty} u(x,t) = \lim_{|x|\to\infty} f(u(x,t)) = 0, \forall t$.

What in (3.1.3a) is meant by sufficiently smooth and what the other conditions are will be specified later on.

For some literature about the problems of existence, uniqueness and regularity, we refer the reader to: [BBM], [BS], [Co], [D], [Lax], [ST], [T 2], [Te], [TM].

The first essential step in solving (3.1.1) with the IST, consists of giving evolution equations for the spectral data. We use the evolution equations as derived in [EvH], Chapter 7.

$$(3.1.4)\qquad \frac{d\lambda_n(t)}{dt} = \varepsilon \int_{-\infty}^{\infty} f(u(x,t))\psi_n^2(x,t)dx \ ; \quad \lambda_n(t) = -\,k_n^2(t) \ .$$

$$(3.1.5)\qquad \frac{\partial b(k,t)}{\partial t} - 8ik^3 b(k,t) = \frac{\varepsilon}{2ik} \int_{-\infty}^{\infty} f(u(x,t))\psi^2(x,k,t)dx \ .$$

$$(3.1.6)\qquad \frac{d\tilde{c}_n(t)}{dt} - 8k_n^3(t)\tilde{c}_n(t) =$$

$$= \frac{\tilde{c}_n(t)}{2k_n(t)} \left\{ \varepsilon \int_{-\infty}^{\infty} f(u(x,t))\tilde{\phi}_n(x,t)\tilde{\psi}_n(x,t)dx - \theta_n(t)\,\frac{d\lambda_n(t)}{dt} \right\} ,$$

with

$$(3.1.7)\qquad \theta_n(t) = \lim_{x\to\infty} \left\{ \int_{-\infty}^{x} (\tilde{\phi}_n(x',t)\tilde{\psi}_n(x',t) - 1)dx' + 2x \right\} .$$

However, they are not a set of evolution equations with which we can perform inverse scattering, because in (3.1.4) we have used the eigenfunction $\psi_n(x,t)$ normalized according to (2.2.44), while the eigenfunction $\tilde{\psi}_n(x,t)$ in (3.1.6) is normalized according to (2.2.42). What we really need is an evolution equation for the normalization coefficient $c_n(t)$ of $\psi_n(x,t)$.
With:

$$(3.1.8)\qquad \gamma_n(t) = \int_{-\infty}^{\infty} \tilde{\psi}_n^2(x,t)dx \ ,$$

it easily follows that:

$$(3.1.9)\qquad \psi_n = \gamma_n^{-\frac{1}{2}}\tilde{\psi}_n \ ; \quad \phi_n = \gamma_n^{\frac{1}{2}}\tilde{\phi}_n \ ; \quad c_n = \gamma_n^{-\frac{1}{2}}\tilde{c}_n \ .$$

Now, it should be obvious that we can derive an evolution equation for $c_n(t)$ if we can find one for $\gamma_n(t)$. In Appendix B.1 we derive an evolution equation for $\gamma_n(t)$ in a way analogous to that presented in the second edition of [EvH]. We include all details because of some unfortunate misprints in [EvH].
The evolution equation for $\gamma_n(t)$ is given by:

(3.1.10) $$\frac{d}{dt}\gamma_n(t) - 8k_n^3(t)\gamma_n(t) = \frac{\varepsilon}{k_n(t)}\gamma_n(t)G_n(t) ,$$

with

(3.1.11) $$G_n(t) = \int_{-\infty}^{\infty} \phi_n\psi_n\left\{\int_{-\infty}^{\infty} f(u)\psi_n^2\,dx \cdot \int_{-\infty}^{x}\psi_n^2\,dx' - \int_{-\infty}^{x} f(u)\psi_n^2\,dx'\right\}dx + $$
$$+ \int_{-\infty}^{\infty}\psi_n^2\left(\int_{-\infty}^{x} f(u)\phi_n\psi_n\,dx'\right)dx +$$
$$- \left(\int_{-\infty}^{\infty} f(u)\psi_n^2\,dx\right)\cdot \int_{-\infty}^{\infty}\psi_n^2\left\{x + \int_{-\infty}^{x}(\phi_n\psi_n - 1)\,dx' + \frac{1}{2k_n}\right\}dx .$$

Using (3.1.4,6,9,10,11) it follows that:

(3.1.12) $$\frac{dc_n(t)}{dt} - 4k_n^3(t)c_n(t) = \frac{\varepsilon c_n(t)}{2k_n(t)} H_n(t) ,$$

with

(3.1.13) $$H_n(t) = - G_n(t) + \int_{-\infty}^{\infty} f(u)\phi_n\psi_n\,dx - \theta_n \int_{-\infty}^{\infty} f(u)\psi_n^2\,dx .$$

We conclude that the set of evolution equations required to use the IST, is given by (3.1.4,5,12).

Remarks (3.1.1):

1°. Of course at $t = 0$, the set of spectral data $S(0)$ is given by the spectral data of the initial potential $u(x,0) = U(x)$.

2°. Convergence of the integral $\int_{-\infty}^{\infty} f(u)\psi^2\,dx$ in formula (3.1.5) follows immediately from (2.2.22), (2.2.33) and (3.1.3c).

3°. The function $\phi_n(x,t)$ is not uniquely defined by (2.2.47), since we can always add a multiple of the eigenfunction $\psi_n(x,t)$ to it. It is easily seen, however, that $G_n(t)$ and $H_n(t)$ do not change under this transformation. In Appendix B.2 it has been proved that, if u satisfies a growth condition of order 1 and $f(u)$ satisfies a growth condition of order 0, then $\theta_n(t)$ and $H_n(t)$ are well-defined.

4°. Although it is not needed for inverse scattering we will given an evolution equation for the transmission coefficients $a(k,t)$ as well. This is done in Appendix B.1.

In the preceding section, we have shown how a solution $u(x,t)$ of (3.1.1) is associated with the set of spectral data:

(3.1.14) $$S(t) = \{\{k_n(t),c_n(t)\}_{n=1,\dots,N};\ b(k,t),\ k \in \mathbb{R}\}\ ,$$

where the spectral data are evolving according to the evolution equations (3.1.4,5,12).

We now define the function $u_s(x,t)$ as follows:

(3.1.15) $u_s(x,t)$ *is the function associated with a solution* $u(x,t)$ *of the pKdV-initial value problem that, when viewed as a potential in the S.E., has the following set of spectral data:*

$$S_s(t) = \{\{k_n(t),c_n(t)\};\ 0\}\ ,$$

where k_n *and* c_n *evolve according to* (3.1.4), *respectively* (3.1.12).

Note that the quantities occurring on the right-hand side of these evolution equations are still given by $f(u)$, ψ_n^2, etc., and not by $f(u_s)$, ψ_{ns}^2, etc. Here the s-indexed quantities are defined in the obvious way. That is, for instance:

(3.1.16) $\psi_{ns}(x,t)$ *is the eigenfunction of the potential* $u_s(x,t)$ *at eigenvalue* $\lambda_n = -k_n^2(t)$, *normalized according to:*

$$\lim_{x\to\infty} \psi_{ns}(x,t)e^{k_n(t)x} = c_n(t)\ ;\qquad \int_{-\infty}^{\infty} \psi_{ns}^2(x,t)dx = 1\ .$$

From the theory given in Theorem (2.2.7), we know that we can write $u_s(x,t)$ and $\psi_{ns}(x,t)$, respectively, in the form (2.2.59), respectively (2.2.60), where k_n and c_n are now time-dependent according to (3.1.4) and (3.1.12).

We also define:

(3.1.17) $$u_c(x,t) = u(x,t) - u_s(x,t)\ .$$

Of course, we can integrate (3.1.4) and (3.1.12), which leads to:

(3.1.18) $$\lambda_n(t) = \lambda_n(0) + \varepsilon \int_0^t \left\{ \int_{-\infty}^{\infty} (f(u)\psi_n^2)(x,t')dx \right\} dt'\ ,$$

$$(3.1.19)\quad c_n(t) = c_n(0)\exp\left\{4\int_0^t k_n^3(t')dt' + \varepsilon\int_0^t \frac{H_n(t')}{2k_n(t')}dt'\right\}.$$

Another useful observation is that we have two different equations for $\gamma_n(t)$, namely, (2.2.52) and (3.1.10). Differentiating (2.2.52) and carrying out some elementary computations leads to:

$$(3.1.20)\quad \frac{d\gamma_n(t)}{dt} = 8k_n^3(t)\gamma_n(t) + \varepsilon\gamma_n(t)R_n(t) + $$
$$- \gamma_n(t)\frac{d}{dt}\left(\frac{k_n(t)}{\pi}\int_0^\infty \frac{\log(1-|b(k,t)|^2)}{k^2+k_n^2(t)}dk\right),$$

where $R_n(t)$ is defined as:

$$(3.1.21)\quad R_n(t) = \frac{1}{2k_n(t)}\int_{-\infty}^\infty f(u(x,t))\phi_n(x,t)\psi_n(x,t)dx +$$
$$+ \frac{1}{2k_n(t)}\left(\frac{1}{k_n(t)} - \theta_n(t)\right)\int_{-\infty}^\infty f(u(x,t))\psi_n^2(x,t)dx +$$
$$+ \sum_{\substack{m=1\\ m\neq n}}^N \left(\frac{k_n(t)}{k_m(t)}\int_{-\infty}^\infty f(u(x,t))\psi_m^2(x,t)dx +\right.$$
$$\left. - \frac{k_m(t)}{k_n(t)}\int_{-\infty}^\infty f(u(x,t))\psi_n^2(x,t)dx\right)\frac{1}{k_n^2(t) - k_m^2(t)}.$$

Matching (3.1.20) to (3.1.9) gives the following equality:

$$(3.1.22)\quad \frac{d}{dt}\frac{k_n(t)}{\pi}\int_0^\infty \frac{\log(1-|b(k,t)|^2)}{k^2+k_n^2(t)}dk = \varepsilon\left(R_n(t) - \frac{G_n(t)}{k_n(t)}\right).$$

This result will be useful for giving estimates on $\int_{-\infty}^\infty u(x,t)dx$ and $\int_{-\infty}^\infty u^2(x,t)dx$. (See equations (2.2.57).)

III.2. Asymptotic behaviour of eigenfunctions $\psi_n(x,t)$; The emergence of solitons

The asymptotic behaviour of the eigenfunctions plays an important role in this analysis. Among other things, we use the asymptotic results to show the emergence of solitons for a potential $u_s(x,t)$ of the form (2.2.59) with scattering data evolving according to (3.1.4,12).
Since the evolution equations of the scattering data are related to the pKdV, the time scales on which solitons emerge and remain separated will depend on ε. We express this property by introducing a *long-time variable*:

(3.2.1) $\tau = \delta(\varepsilon)t$, $\delta(\varepsilon) = o(1)$ *orderfunction such that* $\frac{\varepsilon}{\delta(\varepsilon)} = O(1)$.

Instead of carrying out asymptotics for $t \to \infty$, uniformly in $x \in D$ on certain regions D, we will now work with asymptotics for $\varepsilon \downarrow 0$ uniformly in $\tau \in [0,A]$, $x \in D$. A is a positive constant.
When changing from the variable t to τ, we will not indicate this in our notation. So:

$$k_n(t) = k_n(\tau) \; ; \quad \psi_n(x,t) = \psi_n(x,\tau) \; , \quad \text{etc.}$$

From the evolution equations for the eigenvalues $k_n(\tau)$, we can see that they are subject to an O(1)-change on compacta on the $1/\varepsilon$-timescale, if:

(3.2.2) $\exists$ *constant* C *such that*

$$\left| \int_{-\infty}^{\infty} f(u(x,\tau))\psi_n^2(x,\tau)dx \right| \leq C \; , \quad \textit{for } \tau \in [0,A] \; .$$

This condition is trivially fulfilled because of (3.1.3b).

Throughout this paper we assume that:

(3.2.3) a) *On the timescales under consideration, the number of eigenvalues of a solution of the pKdV initial value problem does not change. So:* $N(\tau) = N(t=0) = N$.

b) $\exists$ *positive constants* M_1, M_2 *and* μ_j *such that on the timescales under consideration:*

$$0 < M_1 \leq k_1(\tau) \leq \ldots \leq k_N(\tau) \leq M_2 \quad \textit{and}$$

$$k_j(\tau) - k_{j-1}(\tau) \geq \mu_j \ , \qquad j = 1,\dots,N-1 \ .$$

Remarks (3.2.1):

1°. The question of what perturbations and initial conditions leave the number of eigenvalues unchanged on finite time intervals is still open. We do have a criterion in the form of conditions on $u(x,t)$ and its eigenfunctions, that guarantees the number of eigenvalues to be invariant on certain time intervals, namely Theorem (2.2.9).

2°. An additional fact concerning (3.2.3b) is that eigenvalue trajectories cannot intersect each other, since the spectrum of the S.E. is non-degenerate.

In Theorem (2.3.1), we saw that with each soliton there was associated a moving coordinate $z_n = x - 4k_n^2 t$. Of course, the relationship between these moving coordinates and the evolution equation for $c_n(t)$ is no coincidence. Therefore, in our case, we replace the expression $4k_n^2 t$ by:

$$(3.2.4) \qquad \varphi_n(\tau) = \frac{4}{\delta(\varepsilon)k_n(\tau)} \int_0^\tau k_n^3(\tau')d\tau' + \frac{\varepsilon}{\delta(\varepsilon)k_n(\tau)} \int_0^\tau \frac{H_n(\tau')}{2k_n(\tau')}\, d\tau' \ .$$

This expression for $\varphi_n(\tau)$ is not very convenient. To simplify it needs further investigation of $G_n(\tau)$ and $H_n(\tau)$.

We expect $\psi_n(x,\tau)$ to be a function of the variables $\bar{x} = x - \varphi(\tau,\varepsilon)$ and τ: $\psi_n(x,\tau) = \bar{\psi}_n(\bar{x},\tau)$. With this in mind, it is obvious that we try to obtain a first idea about the behaviour of $G_n(\tau)$ and $H_n(\tau)$ by introducing the $\bar{x}$ variable in $G_n(\tau)$ and $H_n(\tau)$, with $\varphi(\tau,\varepsilon)$ unspecified.
We define:

$$(3.2.5) \qquad a) \quad G_n^o(\tau) = G_n(\tau) + \left(\int_{-\infty}^{\infty} f(u)\psi_n^2\, dx \right) \int_{-\infty}^{\infty} x\psi_n^2\, dx \ ,$$

$$b) \quad H_n^o(\tau) = H_n(\tau) + \left(\int_{-\infty}^{\infty} f(u)\psi_n^2\, dx \right) \left(\theta_n - \int_{-\infty}^{\infty} x\psi_n^2\, dx \right) .$$

When, in the integrals occurring in the expressions for $G_n^o(\tau)$ and $H_n^o(\tau)$, we replace x by $\bar{x} = x - \varphi(\tau,\varepsilon)$, x' by $\bar{x}' = x' - \varphi$ and $u(x,\tau)$ by $\bar{u}(\bar{x},\tau)$, etc., then the values of these integrals do not change. The other terms in G_n, H_n will change, however, in the following way:

$$(3.2.6)\quad \Big(\int_{-\infty}^{\infty} f(u)\psi_n^2\,dx\Big)\int_{-\infty}^{\infty} x\psi_n^2\,dx =$$

$$= \Big(\int_{-\infty}^{\infty} f(\bar{u})\bar{\psi}_n^2\,d\bar{x}\Big)\int_{-\infty}^{\infty} \bar{x}\bar{\psi}_n^2\,d\bar{x} + \varphi(\tau,\varepsilon)\int_{-\infty}^{\infty} f(\bar{u})\bar{\psi}_n^2\,d\bar{x}\ ,$$

$$\Big(\int_{-\infty}^{\infty} f(u)\psi_n^2\,dx\Big)\Big(\theta_n - \int_{-\infty}^{\infty} x\psi_n^2\,dx\Big) =$$

$$= \Big(\int_{-\infty}^{\infty} f(\bar{u})\bar{\psi}_n^2\,d\bar{x}\Big)\Big\{\Big(\lim_{\bar{x}\to\infty}\int_{-\infty}^{\bar{x}} (\bar{\phi}_n\bar{\psi}_n - 1)\,d\bar{x}' + 2\bar{x}\Big) - \int_{-\infty}^{\infty} \bar{x}\bar{\psi}_n^2\,d\bar{x}\Big\} +$$

$$+ \varphi(\tau,\varepsilon)\int_{-\infty}^{\infty} f(\bar{u})\bar{\psi}_n^2\,d\bar{x}\ .$$

Using boundedness of G_n^o, H_n^o, $\int_{-\infty}^{\infty} \bar{x}\bar{\psi}_n^2\,d\bar{x}$ and $\lim_{\bar{x}\to\infty}\int_{-\infty}^{\bar{x}} (\bar{\phi}_n\bar{\psi}_n - 1)d\bar{x} + 2\bar{x}$, we can now prove the following lemma.

Lemma (3.2.1):

$$\varphi_n(\tau) = \frac{4}{\delta(\varepsilon)}\int_0^{\tau} k_n^2(\tau')\,d\tau' + O\Big(\frac{\varepsilon\tau}{\delta(\varepsilon)}\Big)\ ,\qquad \tau\in[0,A]\ .$$

Proof:

We start with deriving an estimate for $\psi_n(x,\tau)$. We have:

$$\psi_n(x,\tau) = d_n(\tau)\psi_r(x,ik_n,\tau) = c_n(\tau)\psi_\ell(x,ik_n,\tau)\ .$$

So, using (2.2.62) and (3.2.3) we find:

$$(3.2.7)\quad |\psi_n(x,\tau)| \leqq C\min\Big\{d_n(\tau)e^{k_n(\tau)x},\ c_n(\tau)e^{-k_n(\tau)x}\Big\}\ .$$

Introducing $z_n(x,\tau) = x - \varphi_n(\tau,\varepsilon)$ we get:

$$(3.2.8)\quad a)\quad c_n(\tau)e^{-k_n(\tau)x} = c_n(0)e^{-k_n(\tau)z_n(x,\tau)}\ ,$$

$$\text{b)}\quad d_n(\tau) = \gamma_n^{-\frac{1}{2}}(\tau) = \gamma_n^{-\frac{1}{2}}(0)e^{-k_n\varphi_n} \cdot \exp \frac{-\varepsilon}{\delta(\varepsilon)} \int_0^\tau \frac{G_n - H_n}{2k_n}\, d\tau' .$$

With (3.2.5,6) we see that $G_n - H_n$ only contains terms that are bounded. So:

$$(3.2.9)\qquad d_n(\tau)e^{k_n(\tau)x} = d_n(0)e^{O(\frac{\varepsilon\tau}{\delta(\varepsilon)})}\, e^{k_n z_n} .$$

Combining (3.2.7,8a,9) leads to

$$(3.2.10)\qquad |\psi_n(x,\tau)| \leqq Ce^{-k_n|z_n|} .$$

It should be noted for further purposes, that the same bound holds for $\frac{\partial}{\partial x}\psi_n(x,\tau)$. This also follows from (2.6.62) and (3.2.3).

$$(3.2.11)\qquad |\frac{\partial}{\partial x}\psi_n(x,\tau)| \leqq Ce^{-k_n|z_n|} .$$

Now, taking $\bar{x} = z_n$, we find that:

$$(3.2.12)\quad \text{a)}\quad \int_0^\tau \frac{H_n(\tau')}{2k_n(\tau')}\, d\tau' = \int_0^\tau \mathcal{H}_n(\tau')d\tau' - \int_0^\tau \frac{\varphi_n(\tau')}{2k_n(\tau')}\left(\int_{-\infty}^{\infty} f(\bar{u})\bar{\psi}_n^2\, dz_n\right)d\tau' ,$$

with

$$\text{b)}\quad 2k_n\mathcal{H}_n = H_n^o - \left(\int_{-\infty}^{\infty} f(\bar{u})\bar{\psi}_n^2\, d\bar{x}\right)\cdot$$

$$\cdot\left(\left\{\lim_{\bar{x}\to\infty}\int_{-\infty}^{\bar{x}} (\bar{\phi}_n\bar{\psi}_n - 1)d\bar{x}' + 2\bar{x}\right\} - \int_{-\infty}^{\infty} \bar{x}\bar{\psi}_n^2\, d\bar{x}\right) \text{ is bounded on } [0,A].$$

Next, we substitute (3.2.12a) in (3.2.4) and use (3.1.4). This leads to

$$\varphi_n(\tau) = \frac{4}{\delta(\varepsilon)k_n(\tau)}\int_0^\tau k_n^3(\tau')d\tau' + \frac{1}{k_n(\tau)}\int_0^\tau \varphi_n(\tau')\frac{dk_n}{d\tau'}\, d\tau' +$$

$$+ \frac{\varepsilon}{\delta(\varepsilon)k_n(\tau)}\int_0^\tau \mathcal{H}_n(\tau')d\tau'$$

$\Leftrightarrow$

$$\Leftrightarrow k_n\varphi_n = \frac{4}{\delta(\varepsilon)} \int_0^\tau k_n^3 d\tau' + \int_0^\tau \varphi_n \frac{dk_n}{d\tau'} d\tau' + \frac{\varepsilon}{\delta(\varepsilon)} \int_0^\tau H_n(\tau')d\tau'$$

$$\Leftrightarrow \begin{cases} \frac{d}{d\tau}\varphi_n(\tau) = \frac{4}{\delta(\varepsilon)} k_n^2(\tau) + \frac{\varepsilon}{\delta(\varepsilon)} \frac{H_n(\tau)}{k_n(\tau)} \\ \varphi_n(0) = 0 \end{cases}$$

$$\Leftrightarrow \varphi_n(\tau) = \frac{4}{\delta(\varepsilon)} \int_0^\tau k_n^2(\tau')d\tau' + O(\frac{\varepsilon\tau}{\delta(\varepsilon)}) \ .$$

Q.E.D.

An important corollary of the lemma is:

Corollary (3.2.1):

a) *For all positive constants* A *and* $\delta(\varepsilon)$ *with* $\frac{\varepsilon}{\delta(\varepsilon)} = o(1)$, *a positive constant* σ *exists such that:*

(3.2.13) $\varphi_{n+1}(\tau) - \varphi_n(\tau) \geq \sigma \frac{\tau}{\delta(\varepsilon)}$, $n = 0,\dots,N-1$, $\tau \in [0,A]$.

($\varphi_0 \equiv 0$)

b) *If* $\delta(\varepsilon) = \varepsilon$, *then there exist positive constants* A *and* σ, *such that* (3.2.13) *holds.*

Proof:
Trivial, using $k_n(\tau) = k_n(0) + O(\frac{\varepsilon\tau}{\delta(\varepsilon)})$ and (3.2.3).

Corollary (3.2.1) is an important tool in the proof of the following theorem, concerning the behaviour of $\psi_{ns}(x,\tau)$.
In this theorem, we use the following definitions:

Definitions:

(3.2.14) a) $\delta_n^+(t) = \frac{1}{2k_n(t)} \log\left\{\frac{c_n^2(0)}{2k_n(t)} \prod_{i=n+1}^{N} \left(\frac{k_n(t) - k_i(t)}{k_n(t) + k_n(t)}\right)^2\right\}$;

b) $z_n(x,t) = x - \varphi_n(t)$;

c) $\tilde{z}_n(x,t) = z_n(x,t) - \delta_n^+(t)$;

d) $E_n(t) = \{x \in \mathbb{R} \mid \frac{1}{2}(\varphi_{n-1}(t) + \varphi_n(t)) \leq x \leq \frac{1}{2}(\varphi_{n+1}(t) + \varphi_n(t)\}$,

$n = 2,\dots,N-1$;

$E_1(t) = (-\infty, \frac{1}{2}(\varphi_1(t) + \varphi_2(t))]$; $E_N(t) = [\frac{1}{2}(\varphi_N(t) + \varphi_{N-1}(t)),\infty)$;

e) $h_n(x,t) = c_n(t)\psi_{ns}(x,t)e^{-k_n(t)x}$.

<u>Corollary of (3.2.13):</u>

(3.2.15) For $x \in E_n(t)$, $m \leq n-1$: $z_m(x,t) \geq \frac{1}{2}(\varphi_n(t) - \varphi_{n-1}(t)) \geq \sigma t$.

For $x \in E_n(t)$, $m \geq n+1$: $z_m(x,t) \leq \frac{1}{2}(\varphi_n(t) - \varphi_{n+1}(t)) \leq -\sigma t$.

In some parts of our analysis it is $h_n(x,t)$ instead of $\psi_{ns}(x,t)$ that plays a significant role. Therefore, for later convenience, some of the following results will be expressed in terms of $h_n(x,t)$ as well as in terms of $\psi_{ns}(x,t)$.

<u>Theorem (3.2.1):</u>

Let $u_s(x,t)$ *be a potential in the S.E.* (3.1.2) *of the form* (2.2.59), *with scattering data evolving according to* (3.1.4,12). *Let* (3.2.3) *and* (3.2.13) *be satisfied. Then:*

(3.2.16) $\forall k$, $\exists$ *constant* C *such that for* $x \in \mathbb{R}$ *we have:*

$$\left|\frac{\partial^k}{\partial x^k} h_n(x,t)\right| \leq C\left(1 + e^{2k_n z_n}\right)^{-1} , \qquad n = 1,\dots,N ;$$

$$\left|\frac{\partial^k}{\partial x^k} \psi_{ns}^2(x,t)\right| \leq C\left(e^{-k_n z_n} + e^{k_n z_n}\right)^{-2} , \qquad n = 1,\dots,N .$$

(3.2.17) $$\frac{\partial^j}{\partial x^j} h_m(x,t) = \frac{\partial^j}{\partial x^j} h_{mn}(k_n,\dots,k_N,z_n)\ (1 + O(e^{-\alpha t})\ , \quad x \in E_n(t),\ n \leq m .$$

α *is some positive constant. For the functions* h_{mn} *we can give explicit expressions. In the case* $m = n$ *we have:*

(3.2.18) $$\frac{\partial^j}{\partial x^j} \psi_{ms}^2(x,t) = \frac{\partial^j}{\partial x^j} (\tfrac{1}{2}k_m \operatorname{sech}^2 k_m \tilde{z}_m)\ (1 + O(e^{-\alpha t}))\ , \quad x \in E_m(t) .$$

(3.2.19) $\forall k$, $\exists$ *constant* C *such that for* $x \in E_n(t)$ *we have:*

$$\left|\frac{\partial^{k+1}}{\partial t \partial x^k} \psi_{ms}(x,t)\right| \leq Ce^{-k_m z_n}, \qquad \textit{if } m < n,$$

$$\leq Ce^{k_n z_n}\left(1 + e^{2k_n z_n}\right)^{-1}, \qquad \textit{if } m = n,$$

$$\leq Ce^{k_m z_m}, \qquad \textit{if } m > n.$$

Proof:

The starting point for this proof is a set of equations for the eigenfunctions $\psi_{ns}(x,t)$. These equations appear in the derivation of the explicit expressions (2.2.59) and (2.2.60) for u_s and ψ_{ns}, from the Gel'fand-Levitan equation, see [GGKM 2]. We have:

$$\text{(3.2.20)} \quad \psi_{ms}(x,t) + \sum_{n=1}^{N} \frac{c_m(t)c_n(t)e^{-(k_n(t)+k_m(t))x}}{k_n(t) + k_m(t)} \psi_{ns}(x,t) = c_m(t)e^{-k_m(t)x},$$

$m = 1,\ldots,N$.

We can rewrite (3.2.20) as

$$\text{(3.2.21)} \quad (A(t) + D^{-2}(x,t))\vec{h}(x,t) = \vec{1},$$

where

$A(t)$ is the $N \times N$-matrix with coefficients $(k_n(t) + k_m(t))^{-1}$;

$D(x,t)$ is the $N \times N$-matrix with coefficients $\delta_{mn} c_n(t)e^{-k_n(t)x}$;

$\vec{h}(x,t) = (h_1(x,t),\ldots,h_N(x,t))^T$; $\quad \vec{1} = (1,\ldots,1)^T$.

We define:

(3.2.22) $K(x,t) = \det(A + D^{-2})$;

$K_n(x,t)$ is the determinant of the matrix that is obtained by replacing the n-th column in $A + D^{-2}$ by $\vec{1}$.

Using Cramer's rule we get:

$$\text{(3.2.23)} \quad h_m(x,t) = \frac{K_m(x,t)}{K(x,t)}.$$

The following facts can be observed:

Let C be the matrix as defined in (2.2.58), then:

$$C = DAD \quad \text{and} \quad \det(I + C) = (\det D)^2 \det(A + D^{-2}).$$

Now, using $z_n(x,t) = x - \varphi_n(t)$ and Lemma (2.2.1), it follows that:

(3.2.24) a) K is a polynomial in $e^{2k_n(t)z_n(z,t)}$, $n = 1,\dots,N$, with positive coefficients;

b) K_m is a polynomial in $e^{2k_n(t)z_n(x,t)}$, $n = 1,\dots,m-1,m+1,\dots,N$.

Moreover: If a combination of terms $e^{2k_i z_i}$ occurs in K_m, then this same combination, as well as this combination multiplied by $e^{2k_m z_m}$, occurs in K. (Of course, the coefficients are different.)

Now, using (3.2.3b), it follows from (3.2.24) that a constant C exists such that:

$$\left(1+e^{2k_m z_m}\right)K_m \leq C\cdot K \ , \quad \text{for } x \in \mathbb{R}\ ,\ \tau \in [0,A]\ .$$

And so, using (3.2.23), we get that for $x \in \mathbb{R}$:

$$|h_m(x,t)| \leq C\left(1+e^{2k_m z_m}\right)^{-1} \ , \quad m = 1,\dots,N\ .$$

For the first x-derivative of $h_m(x,t)$, we note the following:

$$\frac{\partial h_m}{\partial x}(x,t) = \frac{1}{K}\frac{\partial K_m}{\partial x} - \frac{K_m}{K^2}\frac{\partial K}{\partial x}\ .$$

It is obvious that $\partial K_m/\partial x$, respectively $\partial K/\partial x$ are again polynomials in $e^{2k_i z_i}$ in which the same combination of terms occur as in K_m, respectively K, with exception of the 0-th order term that disappears when we differentiate.
So, following the same reasoning as above, we conclude that there exist constants C_1, $\hat{C}_1$ such that

$$\left(1+e^{2k_m z_m}\right)\frac{\partial K_m}{\partial x} \leq C_1\cdot K \quad \text{and} \quad \left(1+e^{2k_m z_m}\right)K_m\frac{\partial K}{\partial x} \leq \hat{C}_1\cdot K^2\ .$$

This again implies the existence of a constant C, such that:

$$\left|\frac{\partial h_m(x,t)}{\partial x}\right| \leq C\left(1+e^{2k_m z_m}\right)^{-1} \ , \quad m = 1,\dots,N\ .$$

In a completely analogous way, it follows that a constant C exists for every k, such that

$$\left|\frac{\partial^k}{\partial x^k} h_m(x,t)\right| \leq C\left(1 + e^{2k_m z_m}\right)^{-1} , \quad m = 1,\dots,N,\ x \in \mathbb{R} .$$

Finally, using the boundedness of the eigenvalues and (3.2.14e) this leads to the fact that for all k there exists a constant C such that:

$$\left|\frac{\partial^k}{\partial x^k} \psi_{ms}^2(x,t)\right| \leq C\left(e^{-k_m z_m} + e^{k_m z_m}\right)^{-2} , \quad x \in \mathbb{R},\ m = 1,\dots,N .$$

This completes the proof of (3.2.16).

We will now continue with the proof of (3.2.17,18). For this, it is necessary to analyze the functions $K_m(x,t)$ and $K(x,t)$ in more detail. Using (3.2.15), it follows that:

(3.2.25) $$K = (1 + O(e^{-\alpha t})) \prod_{j=1}^{n-1} c_j^{-2}(0) e^{2k_j z_j} \left(D_n + c_n^{-2}(0) e^{2k_n z_n} D_{n+1}\right) ,$$

for $x \in E_n(t)$;

$$K_m = (1 + O(e^{-\alpha t})) \prod_{j=1}^{n-1} c_j^{-2}(0) e^{2k_j z_j} \left(D_{n,m} + (1-\delta_{mn}) c_n^{-2}(0) e^{2k_n z_n} D_{n+1,m}\right) ,$$

for $m \geq n$, $x \in E_n(t)$.

In these expressions, α is some positive constant and

(3.2.26) $$D_n = D_n(t) = \det(k_{ij}) \text{ with } k_{ij} = \frac{1}{k_i + k_j} , \quad i,j = n,\dots,N ;$$

$$D_{n,m} = D_{n,m}(t) = \det(\alpha_{ij}) \text{ with } \begin{cases} \alpha_{ij} = k_{ij}, & i,j = n,\dots,N,\ j \neq m , \\ \alpha_{im} = 1, & i = n,\dots,N . \end{cases}$$

We have the following relations concerning D_n and $D_{n,m}$:

(3.2.27) $$D_n = \frac{1}{2k_m} \prod_{\substack{i=n \\ i\neq m}}^{N} \left(\frac{k_m - k_i}{k_m + k_i}\right) D_{n,m} ;$$

$$D_n = \frac{1}{2k_n} D_{n+1} \prod_{i=n+1}^{N} \left(\frac{k_n - k_i}{k_n + k_i}\right)^2 .$$

From (3.2.23,25) we get:

$$(3.2.28)\quad h_m(x,t) = (1+O(e^{-\alpha t}))\left(\frac{D_n + c_n^{-2}(0)e^{2k_nz_n}D_{n+1}}{D_{n,m} + c_n^{-2}(0)e^{2k_nz_n}D_{n+1,m}}\right)^{-1} ;\ \text{for } x \in E_n(t),\ m \geq n+1 ;$$

$$h_m(x,t) = (1+O(e^{-\alpha t}))\left(\frac{D_m + c_m^{-2}(0)e^{2k_mz_m}D_{m+1}}{D_{m,m}}\right)^{-1} ,\quad \text{for } x \in E_m(t) .$$

With the help of (3.2.27), we can write (3.2.28) as

$$(3.2.29)\quad \text{a)}\quad h_m(x,t) = (1+O(e^{-\alpha t}))\ \cdot$$

$$\cdot\left(\frac{1}{2k_m}\prod_{\substack{i=n+1\\ i\neq m}}^{N}\left(\frac{k_m-k_i}{k_m+k_i}\right)\frac{\frac{1}{2k_n}\prod_{i=n+1}^{N}\left(\frac{k_n-k_i}{k_n+k_i}\right)^2 + c_n^{-2}(0)e^{2k_nz_n}}{\left(\frac{k_m+k_n}{k_m-k_n}\right)\frac{1}{2k_n}\prod_{i=n+1}^{N}\left(\frac{k_n-k_i}{k_n+k_i}\right)^2 + c_n^{-2}(0)e^{2k_nz_n}}\right) ,$$

for $x \in E_n(t)$, $m \geq n+1$;

$$\text{b)}\quad h_m(x,t) = (1+O(e^{-\alpha t}))\ \cdot$$

$$\cdot\left(\frac{1}{2k_m}\prod_{i=m+1}^{N}\frac{k_m-k_i}{k_m+k_i} + c_m^{-2}(0)e^{2k_mz_m}\prod_{i=m+1}^{N}\frac{k_m+k_i}{k_m-k_i}\right)^{-1} ,\quad \text{for } x \in E_m(t) .$$

So, indeed we have

$$(3.2.30)\quad h_m(x,t) = (1+O(e^{-\alpha t}))h_{mn}(k_n,\ldots,k_N,z_n) ,\quad \text{for } x \in E_n(t),\ n \leq m .$$

As in the proof of (3.2.16), from the structure of $K_m(x,t)$ and $K(x,t)$, it is easy to see that the result (3.2.30) can be extended to (3.2.17).
The proof of (3.2.18) is now simple. From (3.2.29b), we get:

$$\psi_{ms}^2 = h_m^2\, c_m^{-2}(0)e^{2k_mz_m} =$$

$$= (1+O(e^{-\alpha t}))c_m^{-2}(0)e^{2k_mz_m}\left(\frac{1}{2k_m}\prod_{i=m+1}^{N}\frac{k_m-k_i}{k_m+k_i} + c_m^{-2}(0)e^{2k_mz_m}\prod_{i=m+1}^{N}\frac{k_m+k_i}{k_m-k_i}\right)^{-2} =$$

$$= (1+O(e^{-\alpha t}))e^{2k_mz_m}\, 2k_m\,\frac{c_m^2(0)}{2k_m}\, e^{-4k_mz_m}\prod_{i=m+1}^{N}\left(\frac{k_m-k_i}{k_m+k_i}\right)^2 \cdot$$

$$\cdot \left(\frac{c_m^2(0)}{2k_m} e^{-2k_m z_m} \prod_{i=m+1}^{N} \left(\frac{k_m - k_i}{k_m + k_i}\right)^2 + 1\right)^{-2} =$$

$$= (1 + O(e^{-\alpha t})) \frac{2k_m e^{-2k_m \tilde{z}_m}}{\left(e^{-2k_m \tilde{z}_m} + 1\right)^2} = (1 + O(e^{-\alpha t})) \tfrac{1}{2}k_m \operatorname{sech}^2 k_m \tilde{z}_m .$$

It remains to prove (3.2.19). From (3.2.21) we have:

$$(3.2.31) \quad \sum_{\ell=1}^{N} a_{m\ell}(x,t) h_\ell(x,t) = 1 ,$$

with

$$a_{m\ell} = \frac{1}{k_m + k_\ell} + \delta_{m\ell} c_m^{-2}(0) e^{2k_m(x-\varphi_m)} .$$

This implies that:

$$(3.2.32) \quad \sum_{\ell=1}^{N} a_{m\ell}(x,t) \frac{\partial}{\partial t} h_\ell(x,t) = - \sum_{\ell=1}^{N} h_\ell(x,t) \frac{\partial}{\partial t} a_{m\ell}(x,t) =$$

$$= \sum_{\ell=1}^{N} \left[\frac{1}{(k_m + k_\ell)^2} \frac{d}{dt} (k_m + k_\ell) + 2\delta_{m\ell} c_m^{-2}(0) \cdot \right.$$

$$\left. \cdot \left(\frac{d}{dt} k_m \varphi_m - x \frac{dk_m}{dt}\right) e^{2k_m(x-\varphi_m)}\right] h_\ell(x,t) .$$

From Lemma (3.2.1) we know that $\varphi_n(t)$ does not grow faster than linearly with t for all n. Therefore:

$$(3.2.33) \quad \varepsilon x = O\left(\frac{\varepsilon}{\delta(\varepsilon)}\right) \quad \text{for } x \in E_n(t) , \quad \tau = \delta(\varepsilon) t \in [0,A] .$$

Since our first goal is to derive estimates for $\frac{\partial}{\partial t} h_m(x,t)$ we have to invert the matrix $(a_{m\ell}) = A + D^{-2}$. This is possible because $K = \det(A + D^{-2}) > 0$. We define:

$$(3.2.34) \quad B = (b_{m\ell}) = (A + D^{-2})^{-1} .$$

Now using (3.1.4) and (3.2.3,4,16,32,33,34) it easily follows that

$$(3.2.35)\quad \left|\frac{\partial}{\partial t} h_m(x,t)\right| \leq C\left\{\varepsilon \sum_{\ell=1}^{N} |b_{m\ell}(x,t)| + \sum_{\ell=1}^{N} |b_{m\ell}(x,t)e^{2k_\ell z_\ell} h_\ell(x,t)|\right\}.$$

The matrix coefficients $b_{m\ell}$ are given by:

$$(3.2.36)\quad b_{m\ell} = (-1)^{m+\ell}\,\frac{K_{\ell m}}{K}\ ,$$

where $K_{\ell m}$ is the determinant of the $N-1 \times N-1$ matrix that remains when the ℓ-th row and the m-th column are omitted from $A + D^{-2}$.
Using (3.2.15), we can estimate $b_{m\ell}$ in exactly the same way as we estimated $h_m(x,t)$ when proving (3.2.16). We get:

$$
\begin{aligned}
(3.2.37)\quad & |b_{m\ell}(x,t)| \leq Ce^{-2k_m z_m} e^{-2k_\ell z_\ell}\ , && m,\ell < n, && x \in E_n(t)\ ,\\
& |b_{mn}(x,t)| \leq Ce^{-2k_m z_m}\left(1+e^{2k_n z_n}\right)^{-1}\ , && m < n, && x \in E_n(t)\ ,\\
& |b_{m\ell}(x,t)| \leq Ce^{-2k_m z_m}\ , && m < n < \ell, && x \in E_n(t)\ .\\
& |b_{n\ell}(x,t)| \leq Ce^{-2k_\ell z_\ell}\left(1+e^{2k_n z_n}\right)^{-1}\ , && \ell < n, && x \in E_n(t)\ ,\\
& |b_{n\ell}(x,t)| \leq C\left(1+e^{2k_n z_n}\right)^{-1}\ , && \ell \geq n, && x \in E_n(t)\ .\\
& |b_{m\ell}(x,t)| \leq Ce^{-2k_\ell z_\ell}\ , && \ell < n < m, && x \in E_n(t)\ ,\\
& |b_{mn}(x,t)| \leq C\left(1+e^{2k_n z_n}\right)^{-1}\ , && m > n, && x \in E_n(t)\ ,\\
& |b_{m\ell}(x,t)| \leq C\ , && m,\ell > n, && x \in E_n(t)\ .
\end{aligned}
$$

We also need bounds for

$$B_{m\ell}(x,t) := b_{m\ell}\, e^{2k_\ell z_\ell} h_\ell\ .$$

Using (3.2.15,16,37), we get:

$$(3.2.38)\quad |B_{m\ell}(x,t)| \leqq Ce^{-2k_m z_m} e^{-2k_\ell z_\ell}, \qquad \ell,m < n, \quad x \in E_n(t),$$

$$|B_{mn}(x,t)| \leqq Ce^{-2k_m z_m} e^{-2k_n |z_n|}, \qquad m < n, \quad x \in E_n(t),$$

$$|B_{m\ell}(x,t)| \leqq Ce^{-2k_m z_m} e^{2k_\ell z_\ell}, \qquad m < n < \ell, \; x \in E_n(t).$$

$$|B_{n\ell}(x,t)| \leqq Ce^{-2k_\ell z_\ell}\left(1 + e^{2k_n z_n}\right)^{-1}, \qquad \ell < n, \quad x \in E_n(t),$$

$$|B_{nn}(x,t)| \leqq Ce^{-2k_n |z_n|}, \qquad x \in E_n(t),$$

$$|B_{n\ell}(x,t)| \leqq Ce^{2k_\ell z_\ell}\left(1 + e^{2k_n z_n}\right)^{-1}, \qquad \ell > n, \quad x \in E_n(t).$$

$$|B_{m\ell}(x,t)| \leqq Ce^{-2k_\ell z_\ell}, \qquad \ell < n < m, \; x \in E_n(t),$$

$$|B_{mn}(x,t)| \leqq Ce^{-2k_n |z_n|}, \qquad m > n, \quad x \in E_n(t),$$

$$|B_{m\ell}(x,t)| \leqq Ce^{2k_\ell z_\ell}, \qquad \ell,m > n, \quad x \in E_n(t).$$

From (3.2.35,37,38) we conclude:

$$(3.2.39)\quad \left|\frac{\partial}{\partial t} h_m(x,t)\right| \leqq C\left\{\varepsilon\, e^{-2k_m z_m} + e^{-2k_m z_m} e^{-2k_n |z_n|}\right\}, \quad m < n, \; x \in E_n(t),$$

$$\left|\frac{\partial}{\partial t} h_n(x,t)\right| \leqq C\left\{\varepsilon\left(1 + e^{2k_n z_n}\right)^{-1} + e^{-2k_n |z_n|}\right\}, \qquad x \in E_n(t),$$

$$\left|\frac{\partial}{\partial t} h_m(x,t)\right| \leqq C\left\{\varepsilon + e^{-2k_n |z_n|}\right\}, \qquad m > n, \; x \in E_n(t).$$

The relationship between $\frac{\partial}{\partial t}\psi_{ms}$ and $\frac{\partial}{\partial t} h_m$ is given by:

$$(3.2.40)\quad \frac{\partial}{\partial t}\psi_{ms}(x,t) = c_m^{-1}(0)\,\frac{\partial}{\partial t} h_m(x,t) e^{k_m(t)(x-\varphi_m(t))} =$$

$$= c_m^{-1}(0)\left[\frac{\partial}{\partial t} h_m + \left(-4k_m^3 - \varepsilon\,\frac{H_m}{2k_m} + x\,\frac{dk_m}{dt}\right) h_m\right] e^{k_m z_m}.$$

So finally, from (3.2.15,16,39,40), we conclude that:

(3.2.41) For $x \in E_n(t)$, we have

$$\left|\frac{\partial}{\partial t}\psi_{ms}(x,t)\right| \leq \begin{cases} Ce^{-k_m z_m} & \text{if } m < n\ , \\ Ce^{k_n z_n}\left(1+e^{2k_n z_n}\right)^{-1} & \text{if } m = n\ , \\ Ce^{k_m z_m} & \text{if } m > n\ . \end{cases}$$

Again, (3.2.41) can be extended to estimates on $(\partial^{k+1}/\partial x^k \partial t)\psi_{ms}(x,t)$ without difficulty. We will demonstrate how such an extension can be made.
We define:

$$P_m(x,t) = \frac{1}{\varepsilon}\sum_{\ell=1}^{N} \frac{\frac{d}{dt}(k_m + k_\ell)}{(k_m + k_\ell)^2}\, h_\ell(x,t)\ .$$

Note that, with (3.1.4) and (3.2.3,16) it follows that

$$\left|\frac{\partial^k}{\partial x^k} P_m(x,t)\right| \leq C\left(1+e^{2k_m z_m}\right)^{-1}\ .$$

Now, equation (3.2.32) can be written as

$$\sum_{\ell=1}^{N} a_{m\ell}\frac{\partial}{\partial t} h_\ell = \varepsilon P_m + \left(8k_m^3 + \varepsilon\frac{H_m}{k_m} - 2x\frac{d}{dt}k_m\right)c_m^{-2}(0)e^{2k_m z_m} h_m\ .$$

Inverting this equation and differentiating it with respect to x gives:

$$\begin{aligned}\frac{\partial^2}{\partial x \partial t} h_m = \varepsilon&\left\{\frac{\partial}{\partial x}P_m \sum_{\ell=1}^{N} b_{m\ell} + P_m \sum_{\ell=1}^{N}\frac{\partial}{\partial x} b_{m\ell}\right\} + \\ &+ \sum_{\ell=1}^{N}\left\{\frac{\partial}{\partial x} b_{m\ell}\left(8k_\ell^3 + \varepsilon\frac{H_\ell}{2k_\ell} - 2x\frac{d}{dt}k_\ell\right)c_\ell^{-2}(0)h_\ell e^{2k_\ell z_\ell} + \right. \\ &+ b_{m\ell}\left(2k_\ell\left[8k_\ell^3 + \varepsilon\frac{H_\ell}{2k_\ell} - 2x\frac{d}{dt}k_\ell\right] - 2\frac{d}{dt}k_\ell\right)c_\ell^{-2}(0)h_\ell e^{2k_\ell z_\ell} + \\ &\left. + b_{m\ell}\left(8k_\ell^3 + \varepsilon\frac{H_\ell}{2k_\ell} - 2x\frac{d}{dt}k_\ell\right)c_\ell^{-2}(0)\, e^{2k_\ell z_\ell}\frac{\partial h_\ell}{\partial x}\right\}\ .\end{aligned}$$

So it follows that:

$$(3.2.42)\quad \left|\frac{\partial^2}{\partial x\partial t} h_m(x,t)\right| \leq C\left\{\varepsilon \sum_{\ell=1}^{N} \left(|b_{m\ell}(x,t)| + \left|\frac{\partial}{\partial x} b_{m\ell}(x,t)\right|\right) + \right.$$

$$+ \sum_{\ell=1}^{N} \left(|b_{m\ell}(x,t)| + \left|\frac{\partial}{\partial x} b_{m\ell}(x,t)\right|\right)|h_\ell(x,t)|e^{2k_\ell z_\ell} +$$

$$\left. + \sum_{\ell=1}^{N} \left|b_{m\ell}(x,t) \frac{\partial}{\partial x} h_\ell(x,t)\right| e^{2k_\ell z_\ell}\right\} .$$

Comparing (3.2.42) with (3.2.35) shows that we get the same estimates for $\frac{\partial^2}{\partial x\partial t} h_m$ as for $\frac{\partial}{\partial t} h_m$, provided that we have the same estimates for $\frac{\partial}{\partial x} b_{m\ell}$ respectively $\frac{\partial}{\partial x} h_m$, as we have for $b_{m\ell}$, respectively h_m.
In an analogous way, we see that the same estimates are found for $\frac{\partial^{k+1}}{\partial x^k\partial t} h_m$ as for $\frac{\partial}{\partial t} h_m$, provided that we have the same estimates for $\frac{\partial^k}{\partial x^k}$ respectively $\frac{\partial^k}{\partial x^k}$, as for $b_{m\ell}$, respectively h_m.

For $h_m(x,t)$, we already know that the above condition is satisfied. That it is satisfied for $b_{m\ell}(x,t)$ as well, is easily seen from the structure of the polynomials in $\exp(2k_i z_i)$: $K_{\ell m}$ and K.
So, we have:

(3.2.43) For $x \in E_n(t)$:

$$\left|\frac{\partial^{k+1}}{\partial x^k\partial t} h_m(x,t)\right| \leq \begin{cases} C\left\{\varepsilon e^{-2k_m z_m} + e^{-2k_m z_m} e^{-2k_n|z_n|}\right\}, & m < n , \\ C\left\{\varepsilon\left(1 + e^{2k_n z_n}\right)^{-1} + e^{-2k_n|z_n|}\right\}, & m = n , \\ C\left\{\varepsilon + e^{-2k_n|z_n|}\right\}, & m > n . \end{cases}$$

Moreover, with $\psi_{ms} = c_m^{-1}(0)h_m e^{k_m z_m}$ we see that

$$(3.2.44)\quad \left|\frac{\partial^{k+1}}{\partial x^k\partial t} \psi_{ms}\right| \leq C \sum_{\ell=0}^{k} \left(\left|\frac{\partial^\ell}{\partial x^\ell} h_m\right| + \left|\frac{\partial^{\ell+1}}{\partial x^\ell\partial t} h_m\right|\right) e^{k_m z_m} .$$

With (3.2.15,16,43,44), we get (3.2.19).

Q.E.D.

Corollaries:

From (3.2.13,16) it follows that

(3.2.45) ∃ *positive constants* α, C *such that*

$$\left|\frac{\partial^k}{\partial x^k} \psi_{ns}^2(x,t)\right| \leq Ce^{-\alpha t} \ , \quad \tau \in [0,A] \ , \quad x \in E_n^c(t) := \mathbb{R} \setminus E_n(t) \ .$$

It is obvious that also

$$(3.2.46) \quad \left|\frac{\partial^k}{\partial x^k} \operatorname{sech}^2 k_n \tilde{z}_n\right| \leq Ce^{-\alpha t} \ , \quad \tau \in [0,A] \ , \quad x \in E_n^c(t) \ ;$$

$$(3.2.47) \quad \left|\frac{\partial^k}{\partial x^k} \operatorname{sech}^2 k_n \tilde{z}_n\right| \leq Ce^{-\alpha t} \ , \quad \tau \in [0,A] \ , \quad x \in E_m(t) \ , \quad n \neq m \ .$$

So with (3.2.18,45,46) we see

$$(3.2.48) \quad \left|\frac{\partial^k}{\partial x^k} (\psi_{ns}^2(x,t) - \tfrac{1}{2}k_n \operatorname{sech}^2 k_n \tilde{z}_n)\right| \leq Ce^{-\alpha t} \ , \quad \tau \in [0,A] \ , \quad x \in \mathbb{R} \ .$$

Moreover, using

$$u_s(x,t) = -4 \sum_{n=1}^{N} k_n(t)\psi_{ns}^2(x,t)$$

with (3.2.47,48) we find:

$$(3.2.49) \quad \text{a)} \quad \left|\frac{\partial^k}{\partial x^k} \left(u_s(x,t) + \sum_{n=1}^{N} 2k_n^2 \operatorname{sech}^2 k_n \tilde{z}_n\right)\right| \leq Ce^{-\alpha t} \ , \quad \tau \in [0,A] \ , \quad x \in \mathbb{R} \ ;$$

$$\text{b)} \quad \left|\frac{\partial^k}{\partial x^k} \ u_s(x,t) + 2k_m^2 \operatorname{sech}^2 k_m \tilde{z}_m)\right| \leq Ce^{-\alpha t} \ , \quad \tau \in [0,A] \ , \quad x \in E_m(t) \ .$$

Remarks:

1°. The choice of $E_n(t)$ is such that

$$\bigcup_{n=1}^{N} E_n(t) = \mathbb{R} \quad \text{and} \quad E_n^o(t) \cap E_m^o(t) = \emptyset \quad \text{for } n \neq m \ .$$

(where A^o stands for the interior of the region A.) This choice, however, is rather arbitrary. All the results still hold if we define:

$$E_n(t) = \{x \in \mathbb{R} \mid \alpha_n(t) + a_n(\varphi_{n-1}(t) - \varphi_n(t) \leq x - \varphi_n(t) \leq$$

$$\leq \tilde{\alpha}_n(t) + \tilde{a}_n(\varphi_{n+1}(t) - \varphi_n(t)\} ,$$

with a_n and $\tilde{a}_n \in (0,1)$ being constants and $\alpha_n(t)$, $\tilde{\alpha}_n(t)$ being arbitrary functions that remain $O(1)$ on $\tau = \delta(\varepsilon)t \in [0,A]$.

2°. We can give estimates that are valid uniformly in (x,τ) on certain regions. Let $\tau \in [s(\varepsilon),A]$, where $s(\varepsilon)$ satisfies:

$$\text{(3.2.50)} \quad \frac{\delta(\varepsilon)}{s(\varepsilon)} = o(1) \ , \ \varepsilon \downarrow 0 \quad \text{and} \quad s(\varepsilon) = o(1) \ , \ \varepsilon \downarrow 0 \ .$$

Then:

$$e^{-\alpha t} \leq e^{-\alpha \frac{s(\varepsilon)}{\delta(\varepsilon)}} = o(1) \ , \ \varepsilon \downarrow 0 \ , \quad \text{uniformly in } \tau \text{ on } [s(\varepsilon),A] \ .$$

So, for instance from (3.2.45), we see that:

$$\text{(3.2.51)} \quad \frac{\partial^k}{\partial x^k} \psi_{ns}^2(x,\tau) = o(1) \ , \ \varepsilon \downarrow 0 \ , \quad \text{uniformly on } E_n^c(\tau) \times [s(\varepsilon),A] \ .$$

3°. Of course all the results also hold for the KdV, i.e. $\varepsilon = 0$. In that case there is no restriction on the time interval, $t \in [0,\infty)$.
The proof of Theorem (3.2.1) is based on the proof of Theorem (2.3.1), see [T 1].
The results (3.2.48,49) can be considered as an extension of Theorem (2.3.1). They express the emergence of solitons for the pKdV.

4°. In the case $N = 1$, the results (3.2.49) naturally simplify to

$$\text{(3.2.52)} \quad u_s(x,t) = -2\frac{d^2}{dx^2} \log\left(1 + \frac{c_1^2}{2k_1} e^{-2k_1 x}\right) = -\ 2k_1^2 \ \text{sech}^2 k_1(x - p_1) \ ,$$

with

$$\text{(3.2.53)} \quad p_1(t) = \frac{1}{2k_1} \log \frac{c_1^2}{2k_1} = \frac{1}{2k_1}\left\{\log \frac{c_1^2(0)}{2k_1} + \log\left[\exp 8 \int_0^t k_n^3 dt' + \right.\right.$$

$$\left.\left. + \ \varepsilon \int_0^t \frac{H_1}{2k_1} dt'\right]\right\} = \frac{1}{2k_1} \log \frac{c_1^2(0)}{2k_1} + \frac{4}{k_1} \int_0^t k_1^3 \, dt' + \frac{\varepsilon}{2k_1} \int_0^t \frac{H_1}{2k_1} dt' \ .$$

CHAPTER IV
APPROXIMATING A POTENTIAL IN THE SCHRÖDINGER EQUATION BY ITS ASSOCIATED SOLITON-POTENTIAL

IV.1. Theorems based on the work of W. Eckhaus and P.C. Schuur

In the previous chapter we have seen what the asymptotic behavious of $u_s(x,t)$ is like. This chapter is dedicated to showing that $u(x,t)$ can in some sense be approximated by $u_s(x,t)$. To do so, we return to the initial value problem for the pKdV and integrate this problem by means of IST.

$$(4.1.1)\qquad \begin{aligned} &u_t - 6uu_x + u_{xxx} = \varepsilon f(u) \\ &u(x,0) = U(x)\ . \end{aligned}$$

The solution of (4.1.1) is given by

$$(4.1.2)\qquad u(x,t) = -\frac{\partial}{\partial x}\beta(0^+,x,t)\ ,$$

where $\beta(y,x,t)$ is the solution of the Gel'fand-Levitan equation:

$$(4.1.3)\qquad \beta(y,x,t) + \Omega(x+y,t) + \int_0^\infty \Omega(x+y+z,t)\beta(z,x,t)dz = 0\ ,$$

with $y > 0$, $x \in \mathbb{R}$, $t > 0$ and

$$(4.1.4)\qquad \Omega(\xi,t) = \Omega_d(\xi,t) + \Omega_c(\xi,t)\ ;$$

$$(4.1.5)\qquad \Omega_d(\xi,t) = 2\sum_{j=1}^{N} c_j^2(t)\ \exp(-2k_j(t)\xi)\ ;$$

$$(4.1.6)\qquad \Omega_c(\xi,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} b(k,t)e^{2ik\xi}\,dk\ ,$$

and the spectral data evolve according to (3.1.4,12).
Note that in (4.1.3) y is the variable and x and t are parameters.

Even for potentials satisfying the KdV, we cannot solve (4.1.3) explicitly for $b(k,t) \not\equiv 0$. Therefore, some of the problems we encounter here, are similar to the problems that occur when deriving asymptotic estimates for $u(x,t) - u_s(x,t)$ in the case of the KdV itself. An analysis of this asymptotic behaviour has been carried out by Tanaka, [T 3], and, more rigorously, with better results, by Eckhaus and Schuur, [ES], [S]. In the following section we will give an outline of the work of Eckhaus and Schuur, and adapt it to our needs.

Let V be the Banach space of real continuous bounded functions on $(0,\infty)$, equipped with the supremum norm. For each $g \in V$ we define the mappings

$$(4.1.7) \qquad (T_d g)(y) = \int_0^\infty \Omega_d(x+y+z,t)g(z)dz \ ,$$

$$(4.1.8) \qquad (T_c g)(y) = \int_0^\infty \Omega_c(x+y+z,t)g(z)dz \ .$$

Note that T_d clearly is a mapping of V into V. The problem is to find $\beta \in V$ such that:

$$(4.1.9) \qquad (I + T_d)\beta + T_c\beta = -\Omega \ ,$$

$$(4.1.10) \quad \Omega = \Omega_d + \Omega_c \ .$$

We know the solution β_d of

$$(4.1.11) \quad (I + T_d)\beta_d = -\Omega_d \ ,$$

which produces the pure N-soliton solution of the KdV equation with the aid of the formula:

$$(4.1.12) \quad u_s(x,t) = -\frac{\partial}{\partial x}\beta_d(0^+,x,t) \ .$$

By imposing suitable conditions on $b(k,0)$, it is shown that:

$$(4.1.13) \quad |\Omega_c(x+y,t)| + \left|\frac{d}{dx}\Omega_c(x+y,t)\right| \leqq H(y,t) \ , \quad t \geqq t_0, \ \bar{x} = x-vt \geqq -M,$$

$$v > 0 \ , \text{ with}$$

(4.1.14) a) $H(y,t)$ is a monotonically decreasing function of y for fixed t, and

b) $$\sigma(t) := \int_0^\infty H(z,t)dz + \sup_{0<y<\infty} H(y,t) < \infty ,$$

c) $\sigma(t) \to 0$ as $t \to \infty$.

The integrability of $H(y,t)$ is used to show that T_c is a continuous mapping of V into V with $\|T_c\| \to 0$ for $t \to \infty$:

$$(4.1.15) \quad \|T_c g\| \leq \|g\| \int_0^\infty H(z,t)dz \leq \|g\|\sigma(t) .$$

It can be shown that T_c is x-differentiable in V, and, as a final result we get:

$$(4.1.16) \quad \|\Omega_c\|, \|T_c\|, \|\Omega_c'\|, \|T_c'\| \leq \sigma(t) ,$$

where ' means taking the x-derivative.
We now return to the Gel'fand-Levitan equation

$$(4.1.17) \quad (I + T_d)\beta = -(\Omega + T_c\beta) .$$

Since T_d is an integral operator with degenerate kernel, solutions of

$$(4.1.18) \quad (I + T_d)g = f , \quad f,g \in V ,$$

can be studied explicitly. In fact, it is easily seen that the solution of (4.1.18) is given by

$$g(y) = f(y) - \sum_{j=1}^N A_j e^{-2k_j y} ,$$

where the A_j satisfy

$$\sum_{j=1}^N \alpha_{ij} A_j = 2 \int_0^\infty e^{-2k_i z} f(z)dz ,$$

$$\alpha_{ij} = \delta_{ij} c_j^{-2} e^{2k_j x} + \frac{1}{k_i + k_j} .$$

(Note that the matrix $(\alpha_{ij}) = A + D^{-2}$ as defined in (3.2.21).)

So it follows that

$$A_j = 2\sum_{i=1}^{N} b_{ij}\left(\int_0^\infty e^{-2k_i z} f(z)dz\right) , \quad \text{where } (b_{ij}) = (\alpha_{ij})^{-1} .$$

We restate the result above as a lemma. (See [S], Ch. 2, lemma 5.1.)

Lemma (4.1.1):

Let $S := (I + T_d)^{-1}$, *then:*

$$(4.1.19) \quad (Sf)(y) = f(y) - 2\sum_{i,j=1}^{N} b_{ij}\left(\int_0^\infty e^{-2k_i z} f(z)dz\right) e^{-2k_j y} .$$

Corollary:

$$(4.1.20) \quad \|S\| \le a_0 := 1 + \sum_{i,j=1}^{N} \frac{|b_{ij}|}{k_i} ; \quad \|S'\| \le a_1 := \sum_{i,j=1}^{N} \frac{|b'_{ij}|}{k_i} .$$

In [S], the following explicit bounds on b_{ij} and b'_{ij} are given:

$$(4.1.21) \quad \text{a)} \quad |b_{ij}| \le N_{ij} := 2(k_i k_j)^{\frac{1}{2}} \prod_{\substack{\ell=1 \\ \ell \ne i}}^{N} \left|\frac{k_i + k_\ell}{k_i - k_\ell}\right| \prod_{\substack{p=1 \\ p \ne j}}^{N} \left|\frac{k_j + k_p}{k_j - k_p}\right| ;$$

$$\text{b)} \quad |b'_{ij}| \le 2a_0 k_j N_{ij} .$$

(More specific information about bounds on $(\partial^k/\partial x^k)b_{ij}$ can be obtained from the estimates (3.2.37) which also hold for the repeated derivatives of the b_{ij}. Moreover, we have:

$$(4.1.22) \quad \beta_d(y,x,t) = -2\sum_{i,j=1}^{N} b_{ij} e^{-2k_j y}$$

and

$$(4.1.23) \quad \beta'_d(y,x,t) = 4\sum_{\ell,p=1}^{N} k_p b_{\ell p}\left(e^{-2k_p y} - \sum_{i,j=1}^{N} \frac{b_{ij}}{k_j + k_p} e^{-2k_j y}\right) .$$

By inverting (4.1.17) we obtain:

$$(4.1.24) \quad \beta = -S\Omega - ST_c\beta .$$

Now consider the mapping $\tilde{T}$, defined by:

(4.1.25) $\tilde{T}g = f - ST_c g \ , \quad f,g \in V$.

It can easily be proved that (4.1.24) possesses a unique solution $\beta \in V$, by showing that $\tilde{T}$ is a contractive mapping in V: $\|ST_c\| \leq \|S\|\|T_c\| \leq a_0\sigma(t) < 1$ for sufficiently large t. So indeed $\tilde{T}$ is a contractive mapping in V.
An estimate for the solution g of $\tilde{T}g = g$ is given by

(4.1.26) $$\|g\| \leq \frac{1}{1 - \|ST_c\|} \|f\| \ .$$

Now all the estimates needed for the final result are derived. We write:

(4.1.27) $$\beta = \beta_d + \beta_c \ ,$$

with

(4.1.28) $$\beta_d = -S\Omega_d \ .$$

Substitution in (4.1.24) gives

(4.1.29) $$\beta_c + ST_c\beta_c = -S\Omega_c - ST_c\beta_d \ .$$

From the preceding analysis, we know that a unique solution β_c exists.
Using (4.1.16,20,21,26), this solution can be estimated by:

(4.1.30) $$\|\beta_c\| \leq \frac{1}{1 - \|ST_c\|} (\|S\|\|\Omega_c\| + \|ST_c\|\|\beta_d\|) \leq b\cdot\sigma(t) \ ,$$

where b is some constant.
The estimate (4.1.30) is valid for $\bar{x} = x - vt \geq -M$ $(v > 0)$, and for t large enough to let $\tilde{T}$ be a contraction.
Since the solution of the KdV equation is given by

(4.1.31) $$u(x,t) = u_s(x,t) - \frac{\partial}{\partial x} \beta_c(0,x,t) \ ,$$

estimates are needed for $\beta_c'(y,x,t)$.
Differentiating equation (4.1.29) we get:

(4.1.32) $$\beta_c' + ST_c\beta_c' = -S\{T_c'(\beta_c+\beta_d) + \Omega_c' + T_c\beta_d'\} - S'\{\Omega_c + T_c(\beta_c+\beta_d)\} \ .$$

Again, we conclude that for t large enough, a unique solution β_c' exists.
Using (4.1.16,20,21,30,32) we estimate β_c' by

(4.1.33) $\|\beta_c'\| \leqq B\sigma(t)$, where B is some constant, and the estimate is valid on $\bar{x} = x - vt \geqq -M$ $(v > 0)$ and t large enough to let $\tilde{T}$ be a contraction.

From (4.1.31) and (4.1.33) we get the result:

(4.1.34) $|u(x,t) - u_s(x,t)| \leqq B\sigma(t)$ on $\bar{x} = x - vt \geqq -M$, $v > 0$,

and for t large enough.

This concludes the outline of the work of Eckhaus and Schuur.

The following important remarks can be made.

Remarks (4.1.1):

1°. For the KdV, the eigenvalues are constant in time. However, this feature has not been used in the above theory. In fact, if the time evolution of the eigenvalues is such that the 'constants' a_0, a_1 and N_{ij} as defined in (4.1.20,21) remain bounded, then the whole scheme still functions.

2°. The validity of the final result (4.1.34) on the region $\bar{x} = x - vt \geqq -M$ hinges on the fact that the estimate (4.1.13) is valid on that region. If (4.1.13) is valid for $x \in I(t)$, $I(t)$ being a time dependent interval (for instance $[-M+vt,\infty)$), then (4.1.34) is valid for $x \in I(t)$, t large enough.

3°. The condition $\int_0^\infty H(y,t)dy \to 0$, as $t \to \infty$, has been used in the following parts of the theory:

a) To establish the existence and uniqueness of a solution in V of the problems (4.1.29,32), by showing that $\|ST_c\| < 1$ for t large enough.

b) To give an estimate on the solution $g \in V$ of $\tilde{T}g = g$ in (4.1.26). Again, by showing that $\|ST_c\| \leqq a < 1$ for t large enough.
We observe that $\int_0^\infty H(y,t)dy$ does not have to decrease for $t \to \infty$. That is, the results will remain valid, if instead of (4.1.14), we have the weaker properties:

(4.1.35) a) $\infty > \sup_{0<y<\infty} H(y,t) = \sigma(t) \to 0$ as $t \to \infty$;

$$\text{b)} \quad \int_0^\infty H(y,t)dy \leqq \hat{\sigma} < \frac{1}{a_0} \quad \text{for } t \geqq T,\ \hat{\sigma} \text{ a constant.}$$

4°. It is easily seen that if we alter conditions (4.1.14,35a) into

(4.1.36) a) $|\Omega_c(x+y,t)| + |\frac{d}{dx}\,\Omega_c(x+y,t)| \leq H(x+y,t)$, $x \in I(t)$, $t \geq t_0$, $y > 0$;

b) $\infty > \sup_{0<y<\infty} H(x+y,t) = \sigma(x,t)$, $x \in I(t)$, $t \geq t_0$,

then in the final result (4.1.34), we can replace $\sigma(t)$ by $\sigma(x,t)$.

If one knows a priori that (4.1.3) is uniquely solvable in V, then, one can do with weaker conditions on Ω_c. This is the situation we are in, since we are looking for solutions $u(x,t)$ of the pKdV satisfying $u(x,t)$ = [2]. So, we can use the results (2.3.5).

When it is given that (4.1.3) is uniquely solvable, then to estimate $u - u_s$, it is not necessary to start from (4.1.29,32). We can also take the following equation as a starting point:

(4.1.37) $(I + T)\beta_c = -\Omega_c - T_c\beta_d$, $T = T_d + T_c$;

(4.1.38) $(I + T)\beta_c' = -\Omega_c' - T_c'\beta_d - T_c\beta_d' - T'\beta_c$.

It now follows that if:

(4.1.39) $\|(I + T)\beta_c^{(k)}\| = \alpha_k(x,t)\|\beta_c^{(k)}\|$, $k = 0,1$, $\beta_c^{(k)} = \frac{\partial^k}{\partial x^k}\,\beta_c$,
with $\alpha_k(x,t) \geq \alpha_k > 0$,

then, with (4.1.37) and (4.1.38), respectively, we find:

(4.1.40) $\|\beta_c\| \leq \frac{1}{\alpha_0(x,t)}\,(\|\Omega_c\| + \|T_c\beta_d\|)$,

(4.1.41) $\|\beta_c'\| \leq \frac{1}{\alpha_1(x,t)}\,(\|\Omega_c\| + \|T_c'\beta_d\| + \|T_c\beta_d'\| + \|T'\beta_c\|)$.

With (4.1.20,21,22) it is easily seen that the right-hand side of (4.1.40) can be estimated using only (4.1.36). The right-hand side of (4.1.41) can be estimated using (4.1.36) and the following condition:

(4.1.42) $\|T'\beta_c\| \leq C\alpha_1(x,t)\{\|\beta_c\| + \sigma(x,t)\}$.

Summarizing we have: If (4.1.36,39,42) hold, then

(4.1.43) $\|\beta_c'\| \leqq C\sigma(x,t)$.

Of course, the same results can be obtained using (4.1.29,32) instead of (4.1.37,38). Then, however, the conditions (4.1.39,42) must hold for ST_c instead of T.

With regard to conditions (4.1.39,42), we make the following observations.

Observations:

1°. If (4.1.39) holds for ST_c, then it holds for T:

(4.1.44) $I + T = I + T_d + T_c = (I + T_d)(I + ST_c) \Leftrightarrow$

$$\Leftrightarrow I + ST_c = S(I + T) \Rightarrow \|(I + T)\beta_c\| \geqq \frac{1}{a_0} \|(I + ST_c)\beta_c\| \ .$$

2°. From [EvH], § 4.5, we know that in $L_2(0,\infty)$, the equation $(I + T)g = 0$ only has the trivial solution. (Of course, this is part of the proof that (4.1.3) is uniquely solvable in $F(\mathbb{R} \to L_2(0,\infty))$.) The $\alpha_k(x,t)$ in (4.1.39) are therefore well-defined and positive.

3°. (4.1.42) is certainly fulfilled if $\|T'\beta_c\| \leqq C\alpha_1(x,t)\|T\beta_c\|$:

$$T\beta_c = -\Omega_c - T_c\beta_d - \beta_c \Rightarrow$$

$$\Rightarrow \|T\beta_c\| \leqq \|\Omega_c\| + \|T_c\beta_d\| + \|\beta_c\| \leqq \sigma(x,t) + \|\beta_c\| \ .$$

4°. $$ST_c = T \Leftrightarrow T_c = T(I + T_d) \Leftrightarrow (I + T)T_d = 0 \Leftrightarrow T_d = 0 \ .$$

In the last equivalence we have used the second observation. In this case (4.1.37,38) reduce to:

$$(I + T_c)\beta_c = -\Omega_c \ ; \quad (I + T_c)\beta_c' = -\Omega_c' - T_c'\beta_c \ .$$

We can summarize the preceding theory in a theorem.

Theorem (4.1.1):

Let $u(x,t)$ *be a* t-*parameter family of potentials in the time-independent S.E. satisfying* $u(x,t) = [2]$. *Let* $\{\{k_n(t),c_n(t)\}_{n=1,\dots,N};\ b(k,t),\ k \in \mathbb{R}\}$ *be the scattering data of* $u(x,t)$. *Let* $u_s(x,t)$ *be the potential with scattering data* $\{\{k_n(t),c_n(t)\}_{n=1,\dots,N};\ 0\}$.

Suppose that:

1°. *The number* N *of eigenvalues is time-independent and there exist constants* $M_1, M_2, \mu_j > 0$ *such that*

$$0 < M_1 \leqq k_1(t) \leqq \dots \leqq k_N(t) \leqq M_2 , \forall t \geqq 0 ;$$

$$|k_{j+1}(t) - k_j(t)| \geqq \mu_j , \quad j = 1,\dots,N-1 , \forall t \geqq 0 .$$

2°. $$|\Omega_c(x+y,t)| + |\frac{d}{dx}\Omega_c(x+y,t)| \leqq H(x+y,t)$$

for $y > 0$, $t \geqq t_0$ *and* $x \in I(t)$, *where* $I(t)$ *is a time-dependent interval, and*

$$\infty > \sup_{0<y<\infty} H(x+y,t) = \sigma(x,t) , \quad x \in I(t) , \quad t \geqq t_0 .$$

3°. *For* $P = T$ *or* $P = ST_c$ *we have*

$$\|(I+P)\beta_c^{(k)}\| = \alpha_k(x,t)\|\beta_c^{(k)}\| , \quad k = 0,1 , \quad \beta_c^{(k)} = \frac{\partial^k}{\partial x^k}\beta_c ,$$

with $\alpha_k(x,t) \geqq \alpha_k > 0$, $x \in S(t)$, $t \geqq t_0$.

$$\|P'\beta_c\| \leqq C\alpha_1(x,t)\{\|\beta_c\| + \sigma(x,t)\} , \quad x \in I(t) , \quad t \geqq t_0 .$$

Then:

$$|u(x,t) - u_s(x,t)| \leqq C\sigma(x,t) , \quad x \in I(t) , \quad t \geqq t_0 .$$

Remark (4.1.2):
A special case arises if ${}_0\int^{\infty} H(x+y+z,t)dz \leqq \hat{\sigma}(t)$ for $x \in I(t)$, $y > 0$, $t \geqq t_0$, with $\hat{\sigma}(t) \to 0$ as $t \to \infty$. Then $\|T_c\| \leqq \hat{\sigma}(t)$, $\|T_c'\| \leqq \hat{\sigma}(t)$ and the third condition of Theorem (4.1.1) is trivially fulfilled for $P = ST_c$.

In our theory not only do we need an estimate on $u(x,t) - u_s(x,t)$, we also need estimates on the differences $\psi_n(x,t) - \psi_{ns}(x,t)$, $n = 1,\dots,N$.
We emphasize the fact that $\psi_n \neq \psi_{ns}$.
By definition of u_s (see (3.1.15)), the eigenvalues $\lambda_n = -k_n^2$ and normalization coefficients c_n of u_s, are equal to those of u.
$\psi_n(x,t)$ is the eigenfunction at eigenvalue $\lambda_n(t)$ of the potential $u(x,t)$ with

$$\lim_{x\to\infty} \psi_n(x,t)e^{k_n(t)x} = c_n(t) \ .$$

ψ_{ns} is the eigenfunction at eigenvalue λ_n of the potential $u_s(x,t)$ with

$$\lim_{x\to\infty} \psi_{ns}(x,t)e^{k_n(t)x} = c_n(t) \ .$$

The estimates on $\psi_n - \psi_{ns}$ can be expressed in terms of the estimates on $u - u_s$ in the following way:

<u>Theorem(4.1.2)</u>:
Let ψ_{ns} *and* $\phi_{ns}\psi_n$ *be uniformly bounded on the region* $x \in [\alpha(t),\infty)$, $t \geqq T$. (See (3.2.10)) and (2.2.44,48).) *Then:*

$$|\psi_n(x,t) - \psi_{ns}(x,t)| \leqq C \int_x^\infty |u(\xi,t) - u_s(\xi,t)| d\xi \ , \quad x \in [\alpha(t),\infty) \ , \quad t \geqq T \ .$$

<u>Proof</u>:
For ψ_n, ψ_{ns} and ϕ_{ns} we have the following equations:

(1) $\quad \psi_n'' + (u - \lambda_n)\psi_n = 0 \ , \quad ' = \frac{\partial}{\partial x} \ ;$

(2) $\quad \psi_{ns}'' + (u_s - \lambda_n)\psi_{ns} = 0 \ ;$

(3) $\quad \lim_{x\to\infty} \psi_n e^{k_n x} = \lim_{x\to\infty} \psi_{ns} e^{k_n x} = c_n \ ;$

(4) $\quad \lim_{x\to\infty} \psi_n' e^{k_n x} = \lim_{x\to\infty} \psi_{ns}' e^{k_n x} = - k_n c_n \ ;$

(5) $\quad \lim_{x\to\infty} \phi_{ns} e^{-k_n x} = -\frac{1}{c_n} \ , \quad \lim_{x\to\infty} \phi_{ns}' e^{-k_n x} = -\frac{k_n}{c_n} \ .$

We define:

(4.1.45) $\quad v_n(x,t) = \psi_n(x,t) - \psi_{ns}(x,t) \ .$

From the above equations, it follows that v_n is the solution of the following problem:

(4.1.46) a) $v_n'' + (u_s - \lambda_n)v_n = (u_s - u)\psi_n$,

b) $\lim_{x\to\infty} v_n e^{k_n x} = \lim_{x\to\infty} v_n' e^{k_n x} = 0$.

Therefore, v_n is given by

(4.1.47) $$v_n(x,t) = \frac{1}{2k_n(t)} \int_x^\infty [\psi_{ns}(x,t)\phi_{ns}(\xi,t) - \phi_{ns}(x,t)\psi_{ns}(\xi,t)] \cdot$$
$$\cdot [u(\xi,t) - u_s(\xi,t)]\psi_n(\xi,t)d\xi \ .$$

It can easily be seen that (4.1.47) satisfies (4.1.46a) by substitution. The conditions (4.1.46b) are shown to be satisfied as follows:

$$|v_n(x,t)e^{k_n(t)x}| \le \frac{1}{2k_n(t)} e^{k_n(t)x} |\psi_{ns}(x,t)| \int_x^\infty |\phi_{ns}\psi_n||u - u_s|d\xi +$$
$$+ \frac{1}{2k_n(t)} \int_x^\infty |e^{k_n(t)x} \phi_{ns}(x,t)\psi_{ns}(\xi,t)\psi_n(\xi,t)||u(\xi,t) - u_s(\xi,t)|d\xi .$$

Now using the formulas (3) and (5) it is obvious that

$$|v_n(x,t)e^{k_n(t)x}| \le C\cdot c_n(t) \int_x^\infty |u(\xi,t) - u_s(\xi,t)|d\xi \to 0 \quad \text{for } x \to \infty \text{ (t fixed)}.$$

That $\lim_{x\to\infty} v_n' e^{k_n x} = 0$, can be proved analogously.

For $v_n(x,t)$ we have the following estimate:

$$|v_n(x,t)| \le C \int_x^\infty |\psi_{ns}(x,t)||u(\xi,t) - u_s(\xi,t)|d\xi +$$
$$+ C \int_x^\infty |\psi_{ns}(\xi)||u(\xi,t) - u_s(\xi,t)| d\xi \le C \int_x^\infty |u(\xi,t) - u_s(\xi,t)| d\xi \ ,$$

where we have used the boundedness of ψ_{ns} and of $\phi_{ns}\psi_n$.

Q.E.D.

As an example of the use of Theorems (4.1.1,2), we will apply them to a potential that satisfies the KdV-initial value problem, where $u(x,0) = U(x)$ is such that the reflection coefficient $b(k,0)$ satisfies:

(4.1.48) *There exists a constant* $\eta > 0$ *such that*

a) $b(k,0)$ *is analytic on* $0 \leqq \text{Im } k \leqq \eta$ *and*

b) *In that strip* $b(k,0) = o(|k|^2)$, *for* $|k| \to \infty$.

(2.2.39) provides us with sufficient conditions for (4.1.48) to hold.

In [ES] it was proved, by means of contour integration in the complex k-plane, that, if (4.1.48) is satisfied for a solution $u(x,t)$ of the KdV, then we have:

(4.1.49) $$H(x+y,t) = \gamma e^{-2\eta(x+y)} e^{8\eta^3 t} \ , \quad x \in \mathbb{R},\ t \geqq t_0 > 0,\ y > 0,$$

γ a positive constant.
We define:

(4.1.50) $$\bar{x} = x - vt \ , \quad v > 0 \text{ constant.}$$

Now we take η such that:

(4.1.51) $$\alpha := 2\eta(v - 4\eta^2) > 0 \ .$$

Then we have:

(4.1.52) a) $$\bar{\sigma}(\bar{x},t) := \sigma(x,t) = \gamma e^{-2\eta\bar{x}} e^{-\alpha t} \leqq Ce^{-\alpha t} \ ,$$

uniformly on $\bar{x} \geqq -M,\ t \geqq t_0$;

b) $$\Big| \int_0^\infty H(x+y+z,t)\,dz \Big| = \frac{\gamma}{2\eta} e^{-2\eta(x+y)} \leqq Ce^{-\alpha t} \ , \quad \bar{x} \geqq -M,\ y > 0,\ t \geqq t_0 .$$

So, with Theorem (4.1.1) we find:

(4.1.53) $$|\bar{u}(\bar{x},t) - \bar{u}_s(\bar{x},t)| \leqq Ce^{-2\eta\bar{x}} e^{-\alpha t} \ , \quad \bar{x} \geqq -M,\ t \geqq T \ ,$$

and using Theorem (4.1.2) we find:

$$(4.1.54)\quad |\bar{\psi}_n(\bar{x},t)-\bar{\psi}_{ns}(\bar{x},t)| \leqq C\int_{\bar{x}}^{\infty} |\bar{u}(\xi,t)-\bar{u}_s(\xi,t)|\,d\xi \leqq Ce^{-2\eta\bar{x}}\, e^{-\alpha t}\ ,$$

$$\bar{x} \geqq -M\ ,\quad t \geqq T\ .$$

(In [S], P.C. Schuur improves on the estimate (4.1.53) for $u(x,t)-u_s(x,t)$ in the sense that he gives an $O(t^{-1/3})$, $t\to\infty$, estimate for $u(x,t)-u_s(x,t)$ uniformly valid on regions $x \geqq -(\mu+\nu t^{1/3})$, $t \geqq t_0$, where μ, ν and t_0 are nonnegative constants.)

In the preceding part of this section we have presented results that can be used for $t\to\infty$ asymptotics. As explained in § III.2, for our purposes, we need results that can be used for $\varepsilon\to 0$ asymptotics on $\frac{1}{\delta(\varepsilon)}$ -timescales, with $\frac{\varepsilon}{\delta(\varepsilon)} = O(1)$. Therefore, we need to reformulate Theorems (4.1.1,2) in such a way that they can be applied to solutions of the pKdV-initial value problem.

Theorem (4.1.3):

Let $u(x,\tau)$ *be a family of regular potentials in the time-independent S.E. with*

(4.1.55) a) $u(x,\tau) = [2]\ ,\quad \tau \in [0,A]\ .$

b) *The eigenvalues* $k_n(\tau)$ *satisfy condition* (3.2.3), $\tau \in [0,A]$.

c) $|\Omega_c(x+y,\tau)| + |\frac{d}{dx}\,\Omega_c(x+y,\tau)| \leqq \sigma(x,t)\ ,\quad y > 0\ ,\quad \tau \in [m(\varepsilon),A]\ ,\quad x \geqq \alpha(\varepsilon,t)\ .$

d) *For* $P = T$ *or* $P = ST_c$ *we have*

$$\|(I+P)\beta_c^{(k)}\| = \alpha_k(x,\tau)\|\beta_c^{(k)}\|\ ,$$

with $\alpha_k(x,\tau) \geqq \alpha_k > 0\ ,\quad \tau \in [m(\varepsilon),A]\ ,\quad x \geqq \alpha(\varepsilon,\tau)\ ,\quad k = 0,1\ ,$

and

$$\|P'\beta_c\| \leqq C\alpha_1(x,\tau)\{\|\beta_c\| + \sigma(x,\tau)\}\ ,\quad \tau \in [m(\varepsilon),A]\ ,\quad x \geqq \alpha(\varepsilon,\tau)\ .$$

Then:

$$(4.1.56)\quad |u(x,\tau)-u_s(x,\tau)| \leqq C\sigma(x,\tau)\ ,\quad \tau \in [m(\varepsilon),A]\ ,\quad x \geqq \alpha(\varepsilon,\tau)\ .$$

If, moreover, we have

(4.1.57) $\psi_{ns}(x,\tau)$ *and* $\phi_{ns}(x,\tau)\psi_n(x,\tau)$ *are uniformly bounded on* $\tau \in [m(\varepsilon),A]$, $x \geqq \alpha(\varepsilon,\tau)$,

then

$$(4.1.58) \quad |\psi_n(x,\tau) - \psi_{ns}(x,\tau)| \leqq C \int_x^\infty \sigma(\xi,\tau)d\xi \ , \quad \tau \in [m(\varepsilon),A] \ , \quad x \geqq \alpha(\varepsilon,\tau) \ .$$

Remark (4.1.3):
As an analogue of Remark (4.1.2), we have that:
If

$$\int_0^\infty |\Omega_c(x+y+z,\tau)| + |\frac{\partial}{\partial x}\Omega_c(x+y+z,\tau)|dz = o(1) \ ,$$

uniformly on $y > 0$, $x \geqq \alpha(\varepsilon,\tau)$, $\tau \in [m(\varepsilon),A]$,

then $\|T_c\| = o(1)$ and $\|T_c'\| = o(1)$, so that (4.1.55d) is trivially fulfilled.
Of course, (4.1.55d) is also trivially fulfilled under the weaker condition:

$$(4.1.59) \quad \int_0^\infty |\Omega_c(x+y+z,\tau)|dz \leqq a < \frac{1}{a_0} \quad \text{for some constant } a \ ,$$

$$\int_0^\infty |\frac{\partial}{\partial x}\Omega_c(x+y+z,\tau)|dz \leqq C \qquad \text{for some constant } C \ .$$

Both bounds must be valid uniformly on $y > 0$, $x \geqq \alpha(\varepsilon,\tau)$, $\tau \in [m(\varepsilon),A]$.

We will postpone applying Theorem (4.1.3) to solutions of (4.1.1) until Chapter V.
First, in § IV.2, we will give another useful theorem which can be used to find estimates for $u(x,t) - u_s(x,t)$ and $\psi_n(x,t) - \psi_{ns}(x,t)$.

IV.2. Theorems based on the Trace-formula

We re-introduce the Trace-formula:

$$u(x,t) = -4 \sum_{n=1}^{N} k_n(t)\psi_n^2(x,t) - \frac{2i}{\pi} \int_{-\infty}^{\infty} kb(-k,t) \frac{\psi^2(x,k,t)}{|a(k,t)|^2} dk .$$

We define:

(4.2.1) $$u_d(x,t) = -4 \sum_{n=1}^{N} k_n(t)\psi_n^2(x,t) ,$$

(4.2.2) $$r(x,t) = -\frac{2i}{\pi} \int_{-\infty}^{\infty} kb(-k,t) \frac{\psi^2(x,k,t)}{|a(k,t)|^2} dk .$$

So, we have:

(4.2.3) $$u(x,t) = u_d(x,t) + r(x,t) .$$

We also have:

(4.2.4) $$u_s(x,t) = -4 \sum_{n=1}^{N} k_n(t)\psi_{ns}^2(x,t) .$$

Since $\psi_n \neq \psi_{ns}$, we see that in general $u_d \neq u_s$. Although obvious from the above, the fact that $u_d \neq u_s$ remains a point of confusion. This confusion is caused by that fact, that it is quite natural to refer to u_d as well as to u_s, as the reflectionless part of u. We will illustrate the difference between u_d and u_s once again, by pointing at the way by which these potentials are defined.
Starting from the set of spectral data $S = \{\{k_n,c_n\}_{n=1,\dots,N}; b(k), k \in \mathbb{R}\}$ of the potential u, u_s is defined as follows:

> In S replace b(k) by zero. u_s is the potential belonging to the set of spectral data thus obtained: $\{\{k_n,c_n\}_{n=1,\dots,N}; 0\}$.

u_d is defined in the following way:

> First, express u in terms of its set of spectral data S, using the Trace-formula. Then, in this formula, replace b(k) by zero. u_d is equal to the remaining terms in the Trace-formula.

Since the actions: 'Expressing a potential in terms of its set of spectral data' and 'Replacing b(k) by zero', do not commute, we see that $u_d \neq u_s$.

We have the following theorem. (Though this theorem, like Theorem 4.1.3), is applicable to a wide class of potentials u(x,t), we will restrict ourselves to potentials that occur as solutions of the (p)KdV-initial value problem (4.1.1) and state the theorem in (ε,τ)-language.)

Theorem (4.2.1):

Let u(x,t) *be a solution of the (p)KdV-initial value problem. Let* $\delta(\varepsilon)$, A *be such that* (3.2.3) *and* (3.2.13) *are satisfied.*

Take u(x,t) *as the potential in the time-independent S.E. If:*

$$(4.2.5) \qquad \int_{-M+\frac{v\tau}{\delta(\varepsilon)}}^{\infty} |r(x,\tau)|\,dx \leq C\tilde{\sigma}(\tau,\varepsilon) \ ,$$

for M *a positive constant (taken to be large enough, see proof),* C *a positive constant, and* v, v_1 *positive constants with* $0 \leq v\,\frac{\tau}{\delta(\varepsilon)} < v_1\,\frac{\tau}{\delta(\varepsilon)} \leq \varphi_1(\tau,\varepsilon)$.

Then, positive constants ρ, μ, C *exist, such that:*

$$(4.2.6) \qquad \left|\frac{\partial^k}{\partial x^k}\,(\psi_n(x,\tau) - \psi_{ns}(x,\tau))\right| \leq C\sigma(\tau,\varepsilon) \ ,$$

uniformly in x *on* $\mathbb{R}$, $\tau \in [\tau_\mu(\varepsilon),A]$, $n = 1,\ldots,N$, $k \in \{0,1\}$,

where

$$(4.2.7) \qquad \sigma(\tau,\varepsilon) = \max\,\{\tilde{\sigma}(\tau,\varepsilon), \exp(-\rho\,\frac{\tau}{\delta(\varepsilon)})\} \ ,$$

(4.2.8) $\tau_\mu(\varepsilon)$ *taken so that:* $0 \leq \sigma(\tau,\varepsilon) < \mu$ *for* $\tau \geq \tau_\mu(\varepsilon)$.

($\tilde{\sigma}(\tau,\varepsilon)$ *must be such that this is possible.)*

Proof

We start by recalling the bounds we have for the eigenfunctions $\psi_n(x,\tau)$ and $\psi_{ns}(x,\tau)$. (See (3.2.10,11).)

$$(4.2.9)\quad |\psi_n(x,\tau)|,|\tfrac{\partial}{\partial x}\psi_n(x,\tau)|,|\psi_{ns}(x,\tau)|,|\tfrac{\partial}{\partial x}\psi_{ns}(x,\tau)| \leq Ce^{-k_n(\tau)|z_n(x,\tau)|},$$

$$x \in \mathbb{R},\quad \tau \in [0,A].$$

By definiton ψ_n and ψ_{ns} have the same eigenvalues and normalization coefficients. Therefore:

$$(4.2.10)\quad a)\quad \left[\frac{d^2}{dx^2} - (u + k_n^2)\right]\psi_n = \left[\frac{d^2}{dx^2} - (u_s + k_n^2)\right]\psi_{ns} = 0 ,$$

$$b)\quad \lim_{x\to\infty} \psi_n e^{k_n x} = \lim_{x\to\infty} \psi_{ns} e^{k_n x} = c_n ,$$

$$c)\quad \lim_{x\to-\infty} \psi_n' e^{k_n x} = \lim_{x\to-\infty} \psi_{ns}' e^{k_n x} = -k_n c_n .$$

We define:

$$(4.2.11)\quad v_n(x,\tau) = \psi_n(x,\tau) - \psi_{ns}(x,\tau) .$$

By substitution of (4.2.1,2,3,4) in (4.2.10), and subtracing the equations for ψ_n from the equations for ψ_{ns}, we find:

$$(4.2.12)\quad a)\quad \left[\frac{d^2}{dx^2} - (u_s + k_n^2)\right]v_n = r(v_n + \psi_{ns}) - (v_n + \psi_{ns})\sum_{m=1}^{N} 4k_m v_m^2 +$$

$$- (v_n + \psi_{ns})\sum_{m=1}^{N} 8k_m v_m \psi_{ms} ,\quad n = 1,\ldots,N ,$$

$$b)\quad \lim_{x\to\infty} v_n e^{k_n x} = \lim_{x\to\infty} v_n' e^{k_n x} = 0 .$$

From this equation we see that it is more convenient to work with

$$(4.2.13)\quad w_n(x,\tau) := v_n(x,\tau)e^{k_n(\tau)x} .$$

w_n has to satisfy:

$$(4.2.14)\quad a)\quad \frac{\partial^2 w_n}{\partial x^2} - 2k_n \frac{\partial w_n}{\partial x} = \left(u_{1n} + u_{2n}w_n + u_{3n}w_n^2\right)w_n + f_n ,$$

b) $\lim_{x\to\infty} w_n = \lim_{x\to\infty} w_n' = 0$,

with

(4.2.15) $$u_{1n} = u_s - 8k_n \psi_{ns}^2 - 8 \sum_{\substack{m=1\\m\neq n}}^{N} k_m \psi_{ms} v_m - 4 \sum_{\substack{m=1\\m\neq n}}^{N} k_m v_m^2 + r ,$$

$$u_{2n} w_n = - 12k_n \psi_{ns} v_n ,$$

$$u_{3n} w_n^2 = - 4k_n v_n^2 ,$$

$$f_n = \psi_{ns} e^{k_n x} \Big(r - 8 \sum_{\substack{m=1\\m\neq n}}^{N} k_m \psi_{ms} v_m - 4 \sum_{\substack{m=1\\m\neq n}}^{N} k_m v_m^2 \Big) .$$

We define:

(4.2.16) $$U_n(x,\tau) = \frac{1}{2k_n} \big(|u_{1n}| + |u_{2n} w_n| + |u_{3n} w_n^2| \big) ,$$

$$U_n^o(x,\tau) = \int_x^\infty U_n(y,\tau)dy .$$

From $u = [0]_u$ and (3.2.3) it follows that:

(4.2.17) $U_n(x,\tau) \leq C$ and $U_n^o(x,\tau) \leq C$, uniformly in (x,τ) on $\mathbb{R} \times [0,A]$, for some positive constant C.

Then (4.2.9) yields:

(4.2.18) $|w_n(x,\tau)| \leq C \cdot c_n(\tau)$, uniformly in x on $\mathbb{R}$.

By some elementary calculations we see that w_n is a classical solution of (4.2.14) iff:

(4.2.19) $$w_n(x,\tau) = \int_x^\infty \Big(1 - e^{-2k_n(y-x)}\Big) \frac{1}{2k_n} \big([u_{1n} + u_{2n} w_n + u_{3n} w_n^2] w_n + f_n\big)dy .$$

Starting with the bound (4.2.18), by iteration in (4.2.19) with τ fixed, we find that:

(4.2.20) $$|w_n(x,\tau)| \leq \frac{e^{U_n^o(x,\tau)}}{2k_n} \int_x^\infty |f_n(y,\tau)|dy \leq C \int_x^\infty |f_n(y,\tau)|dy , \quad x \in \mathbb{R}, \; \tau \in [0,A] .$$

So for $v_n(x,\tau)$ we have

$$(4.2.21)\quad |v_n(x,\tau)| \leq Ce^{-k_n(\tau)x} \int_x^\infty |\psi_{ns}(y,\tau)|e^{k_n(\tau)y} \cdot$$

$$\cdot \left(|r(y,t)| + 4 \sum_{\substack{m=1\\m\neq n}}^{N} [2k_m(\tau)|\psi_{ms}(y,\tau)||v_m(y,\tau)| + k_m(\tau)v_m^2(y,\tau)]\right) dy \ .$$

We rewrite (4.2.21), using the notation $f(x,\tau) = \bar{f}(z_n,\tau)$, in order to get:

$$(4.2.22)\quad |\bar{v}_n(z_n,\tau)| \leq Ce^{-k_n z_n} \int_{z_n}^\infty |\bar{\psi}_{ns}(y,\tau)|e^{k_n y} \cdot$$

$$\cdot \left(|\bar{r}(y,\tau)| + 4 \sum_{\substack{m=1\\m\neq n}}^{N} [2k_m|\bar{\psi}_{ms}||\bar{v}_m| + k_m\bar{v}_m^2]\right) dy \ .$$

We will use (4.2.22) on regions $z_n \geq -M$. First we consider (4.2.22) for the fastest soliton, so for $n = N$.
Note that, from (4.2.9) and (3.2.13), it follows that:

(4.2.23) There exist positive constants C, ρ such that, for $k \in \{0,1\}$:

$$\left|\frac{\partial^k}{\partial x^k}\psi_m(x,\tau)\right| \leq Ce^{-k_m\alpha}\, e^{-(1+\frac{M_2}{M_1})\rho\tau/\delta(\varepsilon)} \ ,$$

for $z_n \geq \alpha$, $m < n$, $\tau \in [0,A]$, $k \in \{0,1\}$;

$$\left|\frac{\partial^k}{\partial x^k}\psi_m(x,\tau)\right| \leq Ce^{k_m\beta}\, e^{-(1+\frac{M_2}{M_1})\rho\tau/\delta(\varepsilon)} \ ,$$

for $z_n \leq \beta$, $m > n$, $\tau \in [0,A]$, $k \in \{0,1\}$.

Here, M_1, M_2 are the constants used in (3.2.3b).
The factor $(1+\frac{M_2}{M_1})$ in the exponent has been introduced for later convenience.
Moreover, also for later use, we take ρ so small that

$$(4.2.24)\quad 0 < \rho \leq M_1(v_1 - v) \ .$$

Of course, the bounds (4.2.23) also hold for $\psi_{ms}(x,\tau)$, and consequently for $v_n(x,\tau)$ as well.
Now, using (4.2.5,7,9,22,23), we get

$$(4.2.25)\quad |\bar{v}_N(z_N,\tau)| \leqq C\sigma(\tau,\varepsilon)e^{-k_N z_N} \quad \text{for } z_N \geqq -M\,,\quad \tau\in[0,A]\,,$$

where $\sigma(\tau,\varepsilon)$ is as defined in (4.2.7) with ρ as in (4.2.23,24).

We can derive analogous results for $\partial v_n/\partial x$ in the following way. Differentiate (4.2.19) to obtain:

$$(4.2.26)\quad \frac{\partial w_n}{\partial x} = -\int_x^\infty e^{-2k_n(y-x)}\left([u_{1n}+u_{2n}w_n+u_{3n}w_n^2]\,w_n+f_n\right)dy\ .$$

With (4.2.17) and (4.2.20) this leads to:

$$(4.2.27)\quad \left|\frac{\partial w_n}{\partial x}\right| \leqq C\int_x^\infty e^{-2k_n(y-x)}\int_y^\infty |f_n(\xi,\tau)|\,d\xi dy + \int_x^\infty e^{-2k_n(y-x)}|f_n(y,\tau)|\,dy \leqq$$

$$\leqq C\int_x^\infty |f_n(y,\tau)|\,dy\ .$$

Using this inequality for $\partial v_n/\partial x$ we find

$$(4.2.28)\quad \left|\frac{\partial v_n}{\partial x}\right| \leqq Ce^{-k_n x}\int_x^\infty |\psi_{ns}|e^{k_n y}\Big(|r| + 4\sum_{\substack{m=1\\ m\neq n}}^{N} 2k_n|\psi_{ms}||v_m| + k_m v_m^2\Big)\,dy\ .$$

Now, with (4.2.5,23,28), we come to the analogue of (4.2.25):

$$(4.2.29)\quad \left|\frac{\partial}{\partial z_N}\bar{v}_N(z_N,\tau)\right| \leqq C\sigma(\tau,\varepsilon)e^{-k_N z_N}\ ,\ \text{for } z_N\geqq -M\,,\quad \tau\in[0,A]\ .$$

Moreover, from (4.2.9) we have:

$$(4.2.30)\quad \left|\frac{\partial^k}{\partial z_n^k}\bar{v}_n(z_n,\tau)\right| \leqq C\sigma(\tau,\varepsilon)\ ,\ \text{for } z_n \leqq \frac{1}{k_n}\log\sigma(\tau,\varepsilon)\ ,$$

$$\tau\in[0,A]\,,\quad k\in\{0,1\}\,,\quad n=1,\dots,N\,.$$

We now have bounds for $\bar{v}_N(z_N,\tau)$ and $\frac{\partial}{\partial z_N}\bar{v}_N(z_N,\tau)$ that are valid for

$$z_n \in \{(-\infty, \frac{1}{k_n}\log \sigma(\tau,\varepsilon) \cup (-M,\infty)\} .$$

However, we need bounds that are valid on the whole real axis. We can get these bounds by using the following lemma:

Lemma:

Let $\sigma(\tau,\varepsilon)$ *be as defined in* (4.2.7) *with* ρ *as in* (4.2.23,24).
Let $\tau_\mu(\varepsilon)$ *be so that* $0 \leq \sigma(\tau,\varepsilon) < \mu$ *for* $\tau \in [\tau_\mu(\varepsilon),A]$.
Then, there exist constants μ, M *such that if:*

$$\alpha = \frac{1}{k_n(\tau)}\log \sigma(\tau,\varepsilon) , \quad \beta = -M ,$$

$$\left|\frac{\partial^k}{\partial z_n^k}\bar{v}_n(\alpha,\tau)\right| \leq C\sigma(\tau,\varepsilon) \quad \textit{and} \quad \left|\frac{\partial^k}{\partial z_n^k}\bar{v}_n(\beta,\tau)\right| \leq C\sigma(\tau,\varepsilon) ,$$

for $k = 0$, *respectively* $k = 0,1$, *then:*

$$\left|\frac{\partial^k}{\partial z_n^k}\bar{v}_n(z_n,\tau)\right| \leq C\sigma(\tau,\varepsilon) \quad \textit{for } z_n \in [\alpha,\beta] , \quad \tau \in [\tau_\mu(\varepsilon),A] ,$$

for $k = 0$, *respectively* $k = 0,1$.

Proof:

First we will finish the proof of this theorem assuming the lemma to hold. Subsequently the lemma will be proved.

Using (4.2.25,30) and the lemma we get:

$$(4.2.31) \quad |v_N(x,\tau)| \leq C\sigma(\tau,\varepsilon) \quad \text{uniformly in } x \text{ on } \mathbb{R}, \quad \tau \in [\tau_\mu(\varepsilon),A] .$$

By means of this bound on v_N, which is valid on the whole real axis, we can give a bound on v_{N-1}. In fact, using (4.2.5,22,23,31), we find:

$$(4.2.32) \quad |\bar{v}_{N-1}(z_{N-1},\tau)| \leq Ce^{-k_{N-1}z_{N-1}} \int_{z_{N-1}}^{\infty} |\bar{\psi}_{N-1,s}| e^{k_{N-1}y} .$$

$$\cdot\left[|\bar{r}| + 4\sum_{m=1}^{N-2}\left(2k_m|\bar{\psi}_{ms}||\bar{v}_m| + k_m\bar{v}_m^2\right) + 4\left(2k_N|\bar{\psi}_{Ns}| + k_N|\bar{v}_N|\right)|\bar{v}_N|\right] dy \leqq$$

$$\leqq Ce^{-k_{N-1}z_{N-1}}\,\sigma(\tau,\varepsilon)\ ,\quad z_{N-1} \geqq -M\ ,\quad \tau \in [\tau_\mu(\varepsilon),A]\ .$$

Now, using (4.2.30,32) and the lemma, we find:

(4.2.33) $|v_{N-1}(x,\tau)| \leqq C\sigma(\tau,\varepsilon)$ uniformly in x on $\mathbb{R}$, $\tau \in [\tau_\mu(\varepsilon),A]$.

Proceeding in this way leads to:

(4.2.34) $|v_n(x,\tau)| \leqq C\sigma(\tau,\varepsilon)$ uniformly in x on $\mathbb{R}$, $\tau \in [\tau_\mu(\varepsilon),A]$,
$n = 1,\ldots,N$.

Analogously, starting from (4.2.28,29) and using (4.2.30) and the lemma, we find:

(4.2.35) $|\frac{\partial}{\partial x} v_n(x,\tau)| \leqq C\sigma(\tau,\varepsilon)$ uniformly in x on $\mathbb{R}$, $\tau \in [\tau_\mu(\varepsilon),A]$,
$n = 1,\ldots,N$.

This proves the theorem, provided that we can prove the lemma.

Proof of the lemma:

For $\bar{v}_n$, we have the following boundary-value problem:

(4.2.36) $$\begin{cases} \dfrac{\partial^2\bar{v}_n}{\partial z_n^2} - k_n^2\bar{v}_n = h_n(z_n,\tau)\ , \\ \bar{v}_n(\alpha) = \sigma_1\ ,\quad \bar{v}_n(\beta) = \sigma_2\ ,\quad \text{with } |\sigma_i| \leqq C\sigma(\tau,\varepsilon)\ ,\ i = 1,2\ . \end{cases}$$

(4.2.37) $$h_n(z_n,\tau) = \bar{u}_s\bar{v}_n + \bar{r}(\bar{v}_n + \bar{\psi}_{ns}) - (\bar{v}_n + \bar{\psi}_{ns})\sum_{m=1}^{N} 4k_m\bar{v}_m^2 + 8k_m\bar{v}_m\bar{\psi}_{ms}\ .$$

Since,

$$\sigma = \max\left\{\tilde{\sigma}, e^{-\rho\frac{\tau}{\delta(\varepsilon)}}\right\} \geqq e^{-\rho\frac{\tau}{\delta(\varepsilon)}}\ ,$$

we see that for the left boundary α, we have:

(4.2.38) $$\alpha = \frac{1}{k_n}\log\sigma \geqq \frac{1}{k_n}\log e^{-\rho\frac{\tau}{\delta(\varepsilon)}} \geqq -\frac{1}{k_n}\rho\frac{\tau}{\delta(\varepsilon)}\ .$$

Together with (4.2.23) this implies that:

$$(4.2.39)\quad \left|\frac{\partial^k}{\partial z_n^k}\bar{\psi}_m(z_n,\tau)\right| \leq Ce^{-\rho\frac{\tau}{\delta(\varepsilon)}} \quad \text{for } m \neq n\,,\ z_n \in [\alpha,\beta]\,,\ \tau \in [0,A]\,.$$

The boundary conditions in (4.2.36) are satisfied by the linear function ϕ_n defined by:

$$(4.2.40)\quad \phi_n(z_n,\tau) = [(\sigma_1-\sigma_2)z_n + (\alpha\sigma_2-\beta\sigma_1)]\,\frac{1}{\alpha-\beta}\,.$$

We notice that:

$$(4.2.41)\quad |\phi_n(z_n,\tau)| \leq \max\{|\sigma_1|,|\sigma_2|\} \leq C\sigma(\tau,\varepsilon)\,,\quad z_n \in [\alpha,\beta]\,.$$

We now can write the solution $\bar{v}_n$ of (4.2.36) in the form

$$(4.2.42)\quad \bar{v}_n(z_n,\tau) = \phi_n(z_n,\tau) + \chi_n(z_n,\tau)\,,$$

where χ_n has to satisfy:

$$(4.2.43)\quad \begin{cases} \dfrac{\partial^2}{\partial z_n^2}\chi_n - k_n^2\chi_n = k_n + k_n^2\phi_n =: g_n\,, \\ \\ \chi_n(\alpha) = \chi_n(\beta) = 0\,. \end{cases}$$

The Green's function for the problem (4.2.43) is given by:

$$(4.2.44)\quad \Delta\Gamma_\ell(z_n,\xi) = \left[e^{k_n(\xi-2\alpha-2\beta)} - e^{-k_n(\xi+2\beta)}\right]e^{k_nz_n} + \left[e^{-k_n\xi} - e^{k_n(\xi-2\alpha)}\right]e^{-k_nz_n}\,,\quad z_n \geq \xi\,,$$

$$\Delta\Gamma_r(z_n,\xi) = \left[e^{k_n(\xi-2\alpha-2\beta)} - e^{-k_n(\xi+2\alpha)}\right]e^{k_nz_n} + \left[e^{-k_n\xi} - e^{k_n(\xi-2\beta)}\right]e^{-k_nz_n}\,,\quad z_n \leq \xi\,,$$

with

$$\Delta = 2k_n\left(e^{-2k_n\alpha} - e^{-2k_n\beta}\right)\,.$$

Notice that:

$$(4.2.45)\quad |\Gamma_\ell(z_n,\xi)| \leq C\,,\quad \alpha \leq \xi \leq z_n \leq \beta\;;\quad \int_\alpha^{z_n} |\Gamma_\ell(z_n,\xi)|\,d\xi \leq C\,,\quad z_n \in [\alpha,\beta]\,,$$

$$|\Gamma_r(z_n,\xi)| \leq C\,,\quad \alpha \leq z_n \leq \xi \leq \beta\;;\quad \int_{z_n}^{\beta} |\Gamma_r(z_n,\xi)|\,d\xi \leq C\,,\quad z_n \in [\alpha,\beta]\,.$$

For the solution χ_n of (4.2.43) we have:

$$(4.2.46)\quad \chi_n(z_n,\tau) = \int_\alpha^{z_n} \Gamma_\ell(z_n,\xi)g_n(\xi)\,d\xi + \int_{z_n}^{\beta} \Gamma_r(z_n,\xi)g_n(\xi)\,d\xi\;.$$

Because of (4.2.24) we have:

$$z_n \geq \alpha \Rightarrow x \geq \frac{1}{k_n}\log\sigma + \varphi_n \geq -\frac{\rho}{M_1}\frac{\tau}{\delta} + \varphi_n > -M + v\frac{\tau}{\delta}\;,\quad n = 1,\ldots,N\,.$$

So, with (4.2.5), we get:

$$(4.2.47)\quad \int_\alpha^\infty |\bar{r}(y,\tau)|\,dy \leq C\sigma(\tau,\varepsilon)\;,\quad n = 1,\ldots,N\,.$$

Now, using (4.2.9,37,39,41,42,45,46,47), we find:

$$(4.2.48)\quad |\chi_n(z_n,\tau)| \leq C\Big\{\frac{1}{\Delta}\int_\alpha^{z_n}\Big(\Big[e^{k_n(\xi-2\alpha-2\beta)} + e^{-k_n(\xi+2\beta)}\Big]e^{k_n z_n} +$$

$$+\Big[e^{k_n(\xi-2\alpha)} + e^{-k_n\xi}\Big]e^{-k_n z_n}\Big)e^{2k_n\xi}|\chi_n(\xi,\tau)|\,d\xi +$$

$$+\frac{1}{\Delta}\int_{z_n}^{\beta}\Big(\Big[e^{k_n(\xi-2\alpha-2\beta)} + e^{-2k_n(\xi+2\alpha)}\Big]e^{k_n z_n} +$$

$$+\Big[e^{-k_n\xi} + e^{k_n(\xi-2\beta)}\Big]e^{-k_n z_n}\Big)e^{2k_n\xi}|\chi_n(\xi,\tau)|\,d\xi + \sigma(\tau,\varepsilon)\Big\}\;,$$

for $z_n \in [\alpha,\beta]\,,\quad \tau \in [0,A]\,.$

In formula (4.2.48) we have:

(4.2.49) a) $\int_{\alpha}^{z_n} (\ldots\ldots) e^{2k_n\xi} d\xi =$

$$= \frac{1}{3k_n} \left\{ e^{k_n(4z_n-2\alpha-2\beta)} + 3e^{2k_n(z_n-\beta)} + 3 + e^{2k_n(z_n-\alpha)} + \right.$$

$$\left. - 4e^{k_n(\alpha-2\beta+z_n)} - 4e^{k_n(\alpha-z_n)} \right\} \leq$$

$$\leq \frac{1}{3k_n} \left(2e^{2k_n(\beta-\alpha)} + 6 \right) .$$

b) $\int_{z_n}^{\beta} (\ldots\ldots) e^{2k_n\xi} d\xi =$

$$= \frac{1}{3k_n} \left\{ 4e^{k_n(z_n+\beta-2\alpha)} + 4e^{k_n(\beta-z_n)} + \right.$$

$$\left. - e^{k_n(4z_n-2\alpha-2\beta)} - 3e^{2k_n(z_n-\alpha)} - 3 - e^{2k_n(z_n-\beta)} \right\} \leq$$

$$\leq \frac{4}{3k_n} \left(e^{2k_n(\beta-\alpha)} + e^{k_n(\beta-\alpha)} \right) .$$

From (4.2.49) is immediately follows that:

(4.2.50) For $\beta = -M$ and $\sigma \leq \mu$, with M large and μ small enough, we have:

a) $\frac{C}{\Delta} \int_{\alpha}^{z_n} (\ldots\ldots) d\xi \leq C\left(e^{2k_n\beta} + \sigma\right) \leq p < 1$, $z_n \in [\alpha,\beta]$, $\tau \in [\tau_\mu, A]$;

b) $\frac{C}{\Delta} \int_{z_n}^{\beta} (\ldots\ldots) d\xi \leq C\left(e^{2k_n\beta} + \sigma\right) \leq p < 1$, $z_n \in [\alpha,\beta]$, $\tau \in [\tau_\mu, A]$.

We define

(4.2.25) $\|\chi_n(\cdot,\tau)\| = \max_{z_n \in [\alpha,\beta]} |\chi(z_n,\tau)|$.

With (4.2.48,50,51) we get

(4.2.52) $\|\chi_n\| \leq p\|\chi_n\| + C\sigma(\tau,\varepsilon)$ with $p < 1$, $\tau \in [\tau_\mu, A]$.

And so we have:

$$(4.2.53)\quad \|\chi_n\| \leqq \frac{1}{1-p}\, C\sigma(\tau,\varepsilon) \leqq C\sigma(\tau,\varepsilon)\ ,\quad \tau \in [\tau_\mu, A]\,.$$

Now with (4.2.41,42,53) we finally arrive at:

$$(4.2.54)\quad |\bar{v}_n(z_n,\tau)| \leqq C\sigma(\tau,\varepsilon) \quad \text{for } z_n \in [\alpha,\beta]\,,\quad \tau \in [\tau_\mu, A]\,.$$

It remains to prove the bound for $\partial\bar{v}_n/\partial z_n$. To that end, we differentiate (4.2.46), and find the following inequality for $\partial\chi_n/\partial z_n$:

$$(4.2.55)\quad \left|\frac{\partial\chi_n}{\partial z_n}\right| \leqq C\left\{\frac{1}{\Delta}\int_\alpha^{z_n}\left(\left[e^{k_n(\xi-2\alpha-2\beta)} + e^{-k_n(\xi+2\beta)}\right]e^{k_n z_n} + \right.\right.$$

$$+\left.\left[e^{k_n(\xi-2\alpha)} + e^{-k_n\xi}\right]e^{-k_n z_n}\right)e^{2k_n\xi}|\chi_n(\xi,\tau)|\,d\xi +$$

$$+\frac{1}{\Delta}\int_{z_n}^{\beta}\left(\left[e^{k_n(\xi-2\alpha-2\beta)} + e^{-k_n(\xi+2\alpha)}\right]e^{k_n z_n} + \right.$$

$$+\left.\left.\left[e^{-k_n\xi} + e^{k_n(\xi-2\beta)}\right]e^{-k_n z_n}\right)e^{2k_n\xi}|\chi_n(\xi,\tau)|\,d\xi + \sigma(\tau,\varepsilon)\right\} \leqq$$

$$\leqq C(\|\chi_n\| + \sigma(\tau,\varepsilon)) \leqq C\sigma(\tau,\varepsilon)\ ,\quad z_n \in [\alpha,\beta]\,,\quad \tau \in [\tau_\mu, A]\,.$$

Obviously, we also have:

$$\left|\frac{\partial\phi_n}{\partial z_n}\right| \leqq C\sigma(\tau,\varepsilon) \quad \text{for } z_n \in [\alpha,\beta]\ ,$$

therefore, we find:

$$(4.2.56)\quad \left|\frac{\partial\bar{v}_n(z_n,\tau)}{\partial z_n}\right| \leqq C\sigma(\tau,\varepsilon)\ ,\quad z_n \in [\alpha,\beta]\,,\quad \tau \in [\tau_\mu, A]\,.$$

Q.E.D.

Theorem (4.2.1) has the following important corollary:

Corollary (4.2.1):

Let the conditions of Theorem (4.2.1) *be satisfied and let, moreover,* $r(x,\tau)$ *satisfy the following condition:*

(4.2.57) $\left|\frac{\partial^k}{\partial x^k} r(x,\tau)\right| \leq C\sigma_1(\tau,\varepsilon)\,, \quad k = 0,1,\ldots,m\,, \quad \tau \in [\tau_\mu(\varepsilon),A]\,, \quad x \in D(\tau,\varepsilon)$

where D *is an arbitrary region.*
Then:

(4.2.58) $\left|\frac{\partial^k}{\partial x^k} (u(x,\tau) - u_s(x,\tau))\right| \leq C \max\{\sigma(\tau,\varepsilon),\sigma_1(\tau,\varepsilon)\}\,,$

$k = 0,1,\ldots,m\,, \quad \tau \in [\tau_\mu(\varepsilon),A]\,, \quad x \in D(\tau,\varepsilon)\,.$

Proof

We have

$$u(x,\tau) - u_s(x,\tau) = -4\sum_{n=1}^{N} k_n(\psi_n + \psi_{ns})(\psi_n - \psi_{ns}) + r\,,$$

$$u'(x,\tau) - u_s'(x,\tau) = -4\sum_{n=1}^{N} k_n(\psi_n' + \psi_{ns}')(\psi_n - \psi_{ns}) + k_n(\psi_n + \psi_{ns})(\psi_n' - \psi_{ns}') + r'\,,$$

$$u''(x,\tau) - u_s''(x,\tau) = -8\sum_{n=1}^{N} k_n[(\psi_n'^2 + \psi_n\psi_n'') - (\psi_{ns}'^2 + \psi_{ns}\psi_{ns}'')] + r'' =$$

$$= -8\sum_{n=1}^{N} k_n(\psi_n'^2 - \psi_{ns}'^2) + k_n[(u + k_n^2)\psi_n^2 - (u_s + k_n^2)\psi_{ns}^2] + r''\,,$$

etc.

For the higher derivatives of $u - u_s$, we use the S.E. to reduce the higher derivatives of ψ_n and ψ_{ns}, to zeroth and first order derivatives. By doing this, we obtain terms that contain derivatives of u and u_s. However, when deriving a bound on $(\partial^k/\partial x^k)(u - u_s)$, the maximum degree of these derivatives is $k - 2$.

So, starting with the bounds that hold for $(\partial^k/\partial x^k)(u - u_s)$, $k = 0,1$, we obtain the bounds on the higher derivatives by induction.

Q.E.D.

In this corollary, we have seen how, with given bounds on $\psi_n - \psi_{ns}$ and condition (4.2.57), we can derive bounds on $u - u_s$. We can also do the opposite. That is, starting with a bound on $u - u_s$ and condition (4.2.57), we can derive a bound for $\psi_n^2 - \psi_{ns}^2$. We note that from (3.2.15) and (4.2.9) it is already known that:

(4.2.59) *There exist positive constants* C, ρ, *such that*

$$|\psi_n(x,\tau)| \leqq Ce^{-\rho\frac{\tau}{\delta(\varepsilon)}} , \quad |\psi_{ns}(x,\tau)| \leqq Ce^{-\rho\frac{\tau}{\delta(\varepsilon)}} ,$$

for $x \in E_n^c(\tau)$, $\tau \in [0,A]$.

So, it suffices to derive a bound for $\psi_n^2 - \psi_{ns}^2$ that is valid on $E_n(\tau)$.

We have the following theorem:

Theorem (4.2.2):

Let u(x,t) *be a solution of the (p)KdV-initial value problem. Let* δ(ε), A *be so that* (3.2.13) *is satisfied.*

If:

(4.2.60) a) $|u(x,\tau) - u_s(x,\tau)| \leqq C\sigma(\tau,\varepsilon)$ *for* $x \in D(\tau,\varepsilon)$, $\tau \in [m(\varepsilon),A]$,

b) $|r(x,\tau)| \leqq C\sigma(\tau,\varepsilon)$ *for* $x \in D(\tau,\varepsilon)$, $\tau \in [m(\varepsilon),A]$,

then, a positive constant ρ *exists, such that:*

(4.2.61) $$|\psi_n^2(x,\tau) - \psi_{ns}^2(x,\tau)| \leqq \max\left\{\sigma(\tau,\varepsilon), e^{-\rho\frac{\tau}{\delta(\varepsilon)}}\right\} ,$$

for $x \in D(\tau,\varepsilon) \cap E_n(\tau)$, $\tau \in [m(\varepsilon),A]$.

Proof:

(4.2.61) is a trivial consequence of

$$u(x,\tau) - u_s(x,\tau) = -4k_n(\psi_n^2 - \psi_{ns}^2) - 4\sum_{\substack{m=1 \\ m\neq n}}^{N} k_m(\psi_m^2 - \psi_{ms}^2) + r(x,\tau)$$

and formulas (4.2.59,60). Q.E.D.

The combination of (4.2.59) and Theorem (4.2.2) can be used as an alternative for Theorem (4.1.2). And, as a matter of fact, that is what we will do when working on the pKdV. The reason why Theorem (4.2.2) is preferred over Theorem (4.1.2) is that we run up against difficulties when trying to find x-integrable o(1) bounds for $u - u_s$. We will encounter the same difficulties, when trying to establish that $\tilde{\sigma}(\tau,\varepsilon)$ in formula (4.2.5) can be taken to be o(1) uniformly on some interval $\tau \in [m(\varepsilon),A]$. Therefore, the obvious way is to use Theorem (4.1.3) in combination with (4.2.59) and Theorem (4.2.2). All this will be explained in the next chapter, in which we will apply the theorems that have been presented in this chapter, to solutions of the pKdV-initial value problem.

CHAPTER V

APPLYING THE THEOREMS OF CHAPTER IV TO SOLUTIONS OF THE pKdV

Throughout this chapter we will assume:

> u(x,t) is a solution of the pKdV initial value problem.
> Condition (3.2.3) is satisfied.
> $\delta(\varepsilon)$, A are such that (3.2.13) is satisfied.

V.1. Results on $\delta^{-1}(\varepsilon)$-timescales with $\delta(\varepsilon) = \varepsilon^p$, $0 \leqslant p < 1$

In this section, we will apply Theorem (4.1.3) to solutions of the pKdV-initial value problem. We will start with considering the condition (4.1.55c). As before, we will use the long-time variable τ. Moreover, we introduce the moving coordinate:

(5.1.1) $\bar{x} = x - \varphi(\tau,\varepsilon)$, where $\varphi(\tau,\varepsilon)$ is such that:

$$v\tau \leqq \delta(\varepsilon)\varphi(\tau,\varepsilon) \leqq \tilde{v}\tau \ , \quad \tau \in [0,A] \text{ , where } v, \tilde{v} \text{ are positive constants.}$$

When changing from the variable x to $\bar{x}$, the x dependent quantity is given a bar. So $u(x,t) = \bar{u}(\bar{x},t)$, etc.

First, we give the set of evolution equations for the spectral data in integrated form:

$$\lambda_n(\tau) = \lambda_n(0) + \frac{\varepsilon}{\delta(\varepsilon)} \int_0^\tau \int_{-\infty}^{\infty} f(\bar{u}(\bar{x},\tau'))\bar{\psi}_n^2(\bar{x},\tau')d\bar{x}\,d\tau' \ ; \tag{5.1.2}$$

$$c_n(\tau) = c_n(0) \exp\left\{\frac{4}{\delta(\varepsilon)} \int_0^\tau k_n^3(\tau')d\tau' + \omega_n(\tau)\right\} , \tag{5.1.3}$$

with

$$\omega_n(\tau) = \frac{\varepsilon}{\delta(\varepsilon)} \int_0^\tau \frac{H_n(\tau')}{2k_n(\tau')}\,d\tau' \ ; \tag{5.1.4}$$

$$(5.1.5)\qquad b(k,\tau) = b(k,0)e^{8ik^3\frac{\tau}{\delta(\varepsilon)}} + \frac{\varepsilon}{\delta(\varepsilon)}\frac{1}{2ik}\int_0^\tau\int_{-\infty}^{\infty} e^{8ik^3\frac{(\tau-\tau')}{\delta(\varepsilon)}} \cdot$$

$$\cdot\, f(\bar{u}(\bar{x},\tau'))\psi^2(\bar{x},k,\tau')d\bar{x}\,d\tau' =: b(k,0)e^{8ik^3\frac{\tau}{\delta(\varepsilon)}} + \tilde{b}(k,\tau)\ .$$

For $\Omega_c(x+y,\tau) = \Omega_c(\bar{x}+\varphi(\tau)+y,\tau)$ we get:

$$(5.1.6)\qquad \Omega_c(\bar{x}+\varphi(\tau)+y,\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty} b(k,0)e^{8ik^3\frac{\tau}{\delta(\varepsilon)}}\, e^{2ik(\bar{x}+y)}\, e^{2ik\varphi(\tau)}\, dk +$$

$$+ \frac{\varepsilon}{\pi\delta(\varepsilon)}\int_{-\infty}^{\infty}\frac{1}{2ik}\left\{\int_0^\tau e^{8ik^3\frac{(\tau-\tau')}{\delta(\varepsilon)}}\left(\int_{-\infty}^{\infty} f(u)\psi^2\, dx\right)d\tau'\right\} e^{2ik(\bar{x}+y)}\ \cdot$$

$$\cdot\, e^{2ik\varphi(\tau)}\, dk =: \frac{1}{\pi}\int_{-\infty}^{\infty} I\,dk + \frac{\varepsilon}{\pi\delta(\varepsilon)}\int_{-\infty}^{\infty} II\,dk\ .$$

We can see that Ω_c consists of two parts. We start by putting a bound on $\int_{-\infty}^{\infty} I\,dk$. We have:

<u>Lemma (5.1.1)</u>:
Let u(x,0) *satisfy the condition* (4.1.48). *Then, positive constants* C, μ, α *exist so that:*

$$(5.1.7)\qquad \left|\int_{-\infty}^{\infty} I\,dk\right| + \left|\frac{\partial}{\partial\bar{x}}\int_{-\infty}^{\infty} I\,dk\right| \le Ce^{-2\mu(\bar{x}+y)}\, e^{-\alpha\frac{\tau}{\delta(\varepsilon)}}\ ,\quad \tau\in[m\delta(\varepsilon),A]\ ,\ \bar{x}\ge -M(\tau,\varepsilon)\ ,$$

where m *is an arbitrary positive constant and* M *an arbitrary bounded function of* τ, ε.
If condition (4.1.48a) *is satisfied, and moreover:*

$$(5.1.8)\qquad \exists\xi>0\ ,\ \textit{with: } b(k,0) = O(|k|^{-(2+\xi)})\ ,\quad |k|\to\infty$$

on the strip $0 \le \mathrm{Im}\, k \le \eta$,

then (5.1.7) *is valid for* $\tau\in[0,A]$, $\bar{x}\ge -M(\tau,\varepsilon)$.

Proof:

We choose $\mu < \eta$ so that: $\alpha := 2\mu(v - 4\mu^2) > 0$ (v as in (5.1.1)).
We integrate over the rectangle Γ in the complex k-plane with vertices at $\pm\rho$, $\pm\rho + i\mu$.
Obviously, we have:

$$\oint_\Gamma I\,dk = 0 .$$

Along the verticals: $k = \pm\rho + is$, $0 \leqq s \leqq \mu$, we have:

$$\left|\int_0^\mu b(\pm\rho+is,0)e^{8i(\pm\rho+is)^3\frac{\tau}{\delta}}\, e^{2i(\pm\rho+is)(\bar{x}+y+\varphi)}\, i\,ds\right| \leqq$$

$$\leqq e^{-2\mu(\bar{x}+y)}\, e^{-2\mu(\varphi-4\mu^2\frac{\tau}{\delta})} \cdot \int_0^\mu |b(\pm\rho+is,0)|e^{-24\rho^2 s\frac{\tau}{\delta}}\, ds$$

and

$$\int_0^\mu |b(\pm\rho+is,0)|e^{-24\rho^2 s\frac{\tau}{\delta}}\, ds \leqq \frac{C}{\rho^2 m} \max_{0\leqq s\leqq\mu} |b(\pm\rho+is,0)| \to 0$$

for $|\rho| \to \infty$, $\tau \geqq m\delta(\varepsilon)$ if (4.1.48) is satisfied,

$$\int_0^\mu |b(\pm\rho+is,0)|e^{-24\rho^2 s\frac{\tau}{\delta}}\, ds \leqq \mu \max_{0\leqq s\leqq\mu} |b(\pm\rho+is,0)| \to 0$$

for $|\rho| \to \infty$, $\tau \geqq 0$ if (5.1.8) is satisfied.

So

$$(5.1.9) \qquad \int_{-\infty}^{\infty} I(k)dk = \int_{-\infty}^{\infty} I(k+i\mu)dk =$$

$$= \int_{-\infty}^{\infty} b(k+i\mu,0)e^{8i(k+i\mu)^3\frac{\tau}{\delta}}\, e^{2i(k+i\mu)(\bar{x}+y+\varphi)}\, dk ,$$

which gives

$$\left|\int_{-\infty}^{\infty} I\,dk\right| \leqq e^{-2\mu(\bar{x}+y)}\, e^{-\alpha\frac{\tau}{\delta}} \int_{-\infty}^{\infty} |b(k+i\mu,0)|e^{-24\mu k^2\frac{\tau}{\delta}}\, dk .$$

Under the consitions imposed it is evident that

$$\int_{-\infty}^{\infty} |b(k+i\mu,0)|\, e^{-24\mu k^2 \frac{\tau}{\delta}}\, dk \leq C .$$

Since the integrand on the right-hand side of (5.1.9) and its $\bar{x}$-derivative are continuous in $(\bar{x},k) \in \mathbb{R} \times \mathbb{R}$ and $\int_{-\infty}^{\infty} |\frac{\partial}{\partial \bar{x}} I(k+i\mu)|\, dk$ is uniformly convergent in $\bar{x}$ on $\bar{x} \geq -M(\tau,\varepsilon)$, we may interchange differentiation with respect to $\bar{x}$ and integration with respect to k in (5.1.9). Doing this we arrive at:

$$|\frac{\partial}{\partial \bar{x}} \int_{-\infty}^{\infty} I\, dk| \leq e^{-2\mu(\bar{x}+y)}\, e^{-\alpha\frac{\tau}{\delta}} \int_{-\infty}^{\infty} |k+i\mu|\, |b(k+i\mu,0)|\, e^{-24\mu k^2 \frac{\tau}{\delta}}\, dk ,$$

$\bar{x} \geq -M$.

Again, under the conditions imposed, it is evident that:

$$\int_{-\infty}^{\infty} |k+i\mu|\, |b(k+i\mu,0)|\, e^{-24\mu k^2 \frac{\tau}{\delta}}\, dk \leq C .$$

Q.E.D.

For $\int_{-\infty}^{\infty} II dk$ we want to derive a bound in an analogous way. We have the following lemma.

<u>Lemma (5.1.2)</u>:
We assume that, for k = 0, *we are in the generic situation, that is (see* (2.2.36)*):*

(5.1.10) $W(0,\tau) \neq 0 , \quad \tau \in [0,A]$.

Moreover, the following conditions must hold:
There exist $\eta = \eta(\varepsilon)$ *and* $\zeta = \zeta(\varepsilon)$ *with:*

(5.1.11) a) $0 < \eta(\varepsilon) < M_1 \leq k_1(\tau) , \quad \tau \in [0,A]$;

b) $$\sup_{\substack{0 \leq \mathrm{Im}\, k \leq \eta(\varepsilon) \\ \tau \in [0,A]}} \left| \int_{-\infty}^{\infty} f(u(x,\tau))\psi^2(x,k,\tau)\,dx \right| = O\left(\frac{|k|^3}{\log |k|}\right), \quad |k| \to \infty ;$$

c) $\left| \int_{-\infty}^{\infty} f(u(x,\tau))\psi^2(x,k+i\eta,\tau)dx \right| \leqq \zeta(\tau,\varepsilon)$ *uniformly in* k *on* $\mathbb{R}$, $\tau \in [0,A]$;

d) II(k) *is analytic in* k *on the strip* $0 < \mathrm{Im}\, k < \eta$, *and continuous on* $0 \leqq \mathrm{Im}\, k \leqq \eta$.

Then a positive constant C *exists so that:*

$$(5.1.12) \quad \left| \int_{-\infty}^{\infty} \mathrm{II}\,dk \right| + \left| \frac{\partial}{\partial \bar{x}} \int_{-\infty}^{\infty} \mathrm{II}\,dk \right| \leqq C\,\frac{\delta(\varepsilon)\zeta(\tau,\varepsilon)}{\eta^3(\varepsilon)}\, e^{-2\eta(\varepsilon)(\bar{x}+y)} \cdot$$

$$\cdot\, e^{-2\eta(\varepsilon)(\varphi(\tau,\varepsilon)-4\eta^2(\varepsilon)\frac{\tau}{\delta(\varepsilon)})}, \quad \bar{x} \geqq -M(\tau,\varepsilon), \quad \tau \in [0,A].$$

Remark (5.1.1):
From (5.1.10), (2.2.4,19,35,36) and Theorem (2.2.1) it immediately follows that (5.1.11d) is satisfied.

Proof:
Again we integrate over the rectangle Γ in the complex k-plane with vertices at $\pm\rho$, $\pm\rho + i$.
From (5.1.11d) we see that

$$(5.1.14) \quad \oint_{\Gamma} \mathrm{II}\,dk = 0 .$$

Along the verticals: $k = \pm\rho + is$, $0 \leqq s \leqq \eta$, we have:

$$(5.1.15) \quad \left| \int_0^{\eta} \frac{e^{2i(\pm\rho+is)(\bar{x}+y+\varphi)}\, e^{8i(\pm\rho+is)^3\frac{\tau}{\delta}}}{2i(\pm\rho+is)} \cdot \right.$$

$$\left. \cdot \left\{ \int_0^{\tau} e^{-8i(\pm\rho+is)^3\frac{\tau'}{\delta}} \left(\int_{-\infty}^{\infty} f(u(x,\tau'))\psi^2(x,\pm\rho+is,\tau')dx \right) d\tau' \right\} i\,ds \right| \leqq$$

$$\leq \frac{1}{2|\rho|} \int_0^{\eta} e^{-2s(\bar{x}+y+\varphi)}\, e^{-(24\rho^2 s-8s^3)\frac{\tau}{\delta}} \left(\int_0^{\tau} e^{(24\rho^2 s-8s^3)\frac{\tau'}{\delta}}\, d\tau' \right) ds \,\cdot$$

$$\cdot \sup_{\substack{0\leq\tau\leq A\\ 0\leq s\leq\eta}} \left| \int_{-\infty}^{\infty} f(u(x,\tau))\psi^2(x,\pm\rho+is,\tau)\,dx \right| \leq$$

$$\leq C\cdot \frac{\tau}{2|\rho|} \int_0^{\eta} \frac{1-e^{-(24\rho^2 s-8s^3)\frac{\tau}{\delta}}}{(24\rho^2 s-8s^3)\frac{\tau}{\delta}}\, ds \,\cdot$$

$$\cdot \sup_{\substack{0\leq\tau\leq A\\ 0\leq s\leq\eta}} \left| \int_{-\infty}^{\infty} f(u(x,\tau))\psi^2(x,\pm\rho+is,\tau)\,dx \right| .$$

Since:

$$\lim_{x\downarrow 0} \frac{1-e^{-x}}{x} = 1 \quad \text{and} \quad \frac{d}{dx}\frac{1-e^{-x}}{x} < 0 \,, \quad \text{for } x > 0 \,,$$

for ρ large enough we find:

$$(5.1.16) \quad \tau \int_0^{\eta} \frac{1-e^{-(24\rho^2 s-8s^3)\frac{\tau}{\delta}}}{(24\rho^2 s-8s^3)\frac{\tau}{\delta}}\, ds \leq \tau \int_0^{\eta} \frac{1-e^{-16\rho^2 s\frac{\tau}{\delta}}}{16\rho^2 s\frac{\tau}{\delta}}\, ds =$$

$$= \frac{\delta}{16\rho^2} \int_0^{16\rho^2\frac{\tau\eta}{\delta}} \frac{1-e^{-w}}{w}\, dw \quad (\text{with } w = 16\rho^2 s\tfrac{\tau}{\delta}) \leq$$

$$\leq \frac{\delta}{16\rho^2} \left(\int_0^1 dw + \int_1^{16\rho^2\frac{\tau\eta}{\delta}} \frac{1}{w}\, dw \right) = \frac{\delta(\varepsilon)}{16\rho^2} \left(1 + \log \frac{16\rho^2\tau\eta(\varepsilon)}{\delta(\varepsilon)} \right) .$$

Combining (5.1.15,16) we find that:
For fixed τ, ε, the contribution along the verticals $k = \pm\rho + is$, $0 \leq s \leq \eta$, is bounded by:

$$(5.1.17) \quad C\, \frac{\log \rho}{\rho^3} \sup_{\substack{0\leq\tau\leq A\\ 0\leq s\leq\eta}} \left| \int_{-\infty}^{\infty} f(u(x,\tau))\psi^2(x,\pm\rho+is,\tau)\,dx \right| .$$

With (5.1.11b), we see that the bound in (5.1.17) tends to zero as $\rho \to \infty$.
So we find:

$$(5.1.18)\quad \int_{-\infty}^{\infty} II(k)dk = \int_{-\infty}^{\infty} II(k+i\eta)dk = \int_{-\infty}^{\infty} \frac{e^{2i(k+i\eta)(\bar{x}+y+\varphi)}}{2i(k+i\eta)} \cdot$$

$$\cdot \left\{ \int_0^{\tau} e^{8i(k+i\eta)^3\frac{(\tau-\tau')}{\delta}} \left(\int_{-\infty}^{\infty} f(u(x,\tau'))\psi^2(x,k+i\eta,\tau')dx \right) d\tau' \right\} dk \ .$$

Using (5.1.11c,19) we get:

$$(5.1.19)\quad \left| \int_{-\infty}^{\infty} II(k)dk \right| \leq e^{-2\eta(\bar{x}+y)} \, e^{-2\eta\varphi}\cdot\zeta\cdot\delta \int_0^{\infty} \frac{1-e^{-(24\eta k^2-8\eta^3)\frac{\tau}{\delta}}}{(24\eta k^2-8\eta^3)(k^2+\eta^2)^{\frac{1}{2}}} dk \ .$$

By substituting $z = \sqrt{3}\,k$ we get:

$$(5.1.20)\quad \int_0^{\infty} \frac{1-e^{-(24\eta k^2-8\eta^3)\frac{\tau}{\delta}}}{(24\eta k^2-8\eta^3)(k^2+\eta^2)^{\frac{1}{2}}} dk = \frac{1}{\sqrt{3}} \int_0^{\infty} \frac{1-e^{-8\eta(z^2-\eta^2)\frac{\tau}{\delta}}}{8\eta(z^2-\eta^2)(\eta^2+\frac{z^2}{3})^{\frac{1}{2}}} dz <$$

$$< \frac{1}{\sqrt{3}} \int_0^{2\eta} \frac{1-e^{-8\eta(z^2-\eta^2)\frac{\tau}{\delta}}}{8\eta^2(z^2-\eta^2)} dz + \frac{1}{\sqrt{3}} \int_{2\eta}^{\infty} \frac{1-e^{-8\eta(z^2-\eta^2)\frac{\tau}{\delta}}}{8\eta^2(z^2-\eta^2)} dz \ .$$

We have

$$(5.1.21)\quad \int_0^{2\eta} \frac{1-e^{-8\eta(z^2-\eta^2)\frac{\tau}{\delta}}}{8\eta^2(z^2-\eta^2)} dz \leq 2\eta \max_{z\in[0,2\eta]} \frac{1-e^{-8\eta(z^2-\eta^2)\frac{\tau}{\delta}}}{8\eta^2(z^2-\eta^2)} = \frac{e^{8\eta^3\frac{\tau}{\delta}}-1}{4\eta^3}$$

and

$$(5.1.22)\quad \int_{2\eta}^{\infty} \frac{1-e^{-8\eta(z^2-\eta^2)\frac{\tau}{\delta}}}{8\eta^2(z^2-\eta^2)} dz \leq \int_{2\eta}^{\infty} \frac{1}{8\eta^2(z^2-\eta^2)} dz = \frac{\log 3}{16\eta^3} \ .$$

Combining (5.1.19) to (5.1.22) we get:

(5.1.23) There exist a positive constant C, so that:

$$\left| \int_{-\infty}^{\infty} II(k)dk \right| \leq C \, \frac{\delta(\varepsilon)\zeta(\tau,\varepsilon)}{\eta^3(\varepsilon)} \, e^{-2\eta(\varepsilon)(\bar{x}+y)} \, e^{-2\eta(\varepsilon)(\varphi(\tau,\varepsilon)-4\eta^2(\varepsilon)\frac{\tau}{\delta(\varepsilon)})} \ ,$$

$\bar{x} \in \mathbb{R}, \quad y \geq 0, \quad \tau \in [0,A]$.

As in the proof of Lemma (5.1..1), it follows that on regions $\bar{x} \geq -M(\tau,\varepsilon)$, it is allowed to interchange differentiation with respect to $\bar{x}$ and integration with respect to k, so that:

$$\frac{\partial}{\partial \bar{x}} \int_{-\infty}^{\infty} II(k)dk = \frac{\partial}{\partial \bar{x}} \int_{-\infty}^{\infty} II(k+i\eta)dk =$$

$$= \int_{-\infty}^{\infty} e^{2i(k+i\eta)(\bar{x}+y+\varphi)} \; \cdot$$

$$\cdot \left\{ \int_0^{\tau} e^{8i(k+i\eta)^3 \frac{(\tau-\tau')}{\delta}} \left(\int_{-\infty}^{\infty} f(u(x,\tau'))\psi^2(x,k+i\eta,\tau')dx \right) d\tau' \right\} dk \; .$$

This integral can be estimated in a way similar to the one used for estimating the integral in (5.1.18). The only difference in the result is a factor $1/2\eta$ that disappears under the $\bar{x}$-differentiation. This is an improvement when $\eta = o(1)$. So, certainly, the bound (5.1.23) is also valid for $\left|\frac{\partial}{\partial \bar{x}} \int_{-\infty}^{\infty} II(k)dk\right|$.

Q.E.D.

A combination of the Lemmas (5.1.1,2) provides us with a bound for Ω_c that can be used in Theorem (4.1.3).
Of course, the question arises when the conditions imposed in the lemmas are fulfilled, and in the case of condition (5.1.11), for what choice of η, ζ. We can make the following remarks in relation to this question:

Remarks (5.1.2):

1°. The condition $W(0,\tau) \neq 0$, $\tau \in [0,A]$, is not quite satisfactory. Although generically $W(0,\tau) \neq 0$, this condition is certainly not satisfied if we take a reflectionless potential as initial function. However, if:

(5.1.24) For all $\tau \in [0,A]$ with $W(0,\tau) = 0$, there exists $r(\varepsilon)$ such that $II(k,\tau)$ is analytic on $B_{r(\varepsilon)}(0) \setminus \{0\}$ ($B_{r(\varepsilon)}(0)$ is the circle with radius $r(\varepsilon)$ and center 0),

then, we can replace (5.1.10) by the weaker condition:

(5.1.25) $W(0,\tau) \neq 0$ almost everywhere on $\tau \in [0,A]$.

Again, we observe that (5.1.24) is certainly satisfied if $u(x,\tau)$ decays exponentially for $|x| \to \infty$, see (2.2.39).
The proof that indeed the same results hold under condition (5.1.25), is given in Appendix C.

2°. Because of the fact that 'birth' or 'death' of eigenvalues is only possible at times τ_0 for which $W(0,\tau_0) = 0$, see Theorem (2.2.9), the condition (3.2.3a) follows directly from (5.1.10).

3°. It is easy to see that, in (5.1.11b), it is not the asymptotic behaviour in k, but the convergence of the x-integral, that causes the biggest restriction.
We know that:
$\psi(x,k,\tau)$ is analytic in k on $0 < \mathrm{Im}\, k < \eta$ and continuous on $0 \leq \mathrm{Im}\, k \leq \eta$.
Moreover, from (2.2.62,63) we see:

$$|\psi(x,k,\tau)| = |a(k,\tau)R(x,k,\tau)e^{-ikx}| \leq \left(1 + O(\tfrac{1}{|k|})\right) e^{U_0/|k|} e^{x\,\mathrm{Im}\,k} .$$

So (5.1.11b) is certainly satisfied if

$$(5.1.26) \qquad \int_{-\infty}^{\infty} |f(u(x,\tau))| e^{2\eta x}\, dx \quad \text{converges.}$$

This also provides us immediately with an upper bound on $\zeta(\tau,\varepsilon)$.
We have:

$$(5.1.27) \qquad \left| \int_{-\infty}^{\infty} f(u(x,\tau))\psi^2(x,k+i\eta,\tau)dx \right| \leq C \int_{-\infty}^{\infty} |f(u(x,\tau))| e^{2\eta x}\, dx .$$

As it turns out, the above method of contour-integration with the smallest upperbound on $\zeta(\tau,\varepsilon)$ given by (5.1.27), does only give results on rather short timescales. We will illustrate this by showing what we get from (5.1.27) when the solution of the pKdV has an *N-soliton structure*. That is, we take f(u) so that:

$$(5.1.28) \qquad f(u) = \sum_{i=1}^{N} f_i(x - \varphi_i(\tau,\varepsilon)) .$$

We then find

$$(5.1.29) \qquad \int_{-\infty}^{\infty} f(u(x,\tau)) e^{2\eta x}\, dx = O\left(e^{2\eta\varphi_N(\tau,\varepsilon)}\right) .$$

Substituting $\zeta(\tau,\varepsilon) = e^{2\eta\varphi_N(\tau,\varepsilon)}$ in (5.1.12), we get:

$$(5.1.30)\quad \frac{\varepsilon}{\delta}\left(\Big|\int_{-\infty}^{\infty} \mathrm{II}\,dk\Big| + \Big|\frac{\partial}{\partial\bar{x}}\int_{-\infty}^{\infty} \mathrm{II}\,dk\Big|\right) \leq C\,\frac{\varepsilon}{\eta^3}\, e^{-2\eta(\bar{x}+y)}\, e^{-2\eta(\varphi-\varphi_N)}\, e^{8\eta^3\frac{\tau}{\delta}} .$$

Of course, since we do not want to lose information about the soliton structure of the solution, we must take:

$$\varphi(\tau,\varepsilon) \leq \varphi_1(\tau,\varepsilon) .$$

Using (3.2.13), this implies that for $N > 1$ a positive constant σ exists so that

$$\varphi_N(\tau,\varepsilon) - \varphi(\tau,\varepsilon) \geq \sigma\,\frac{\tau}{\delta(\varepsilon)} , \quad \tau \in [0,A] .$$

To avoid explosion of the bound in (5.1.30), we must take $\eta(\varepsilon)$ such that:

$$\frac{\eta(\varepsilon)}{\delta(\varepsilon)} = O(1) , \quad \varepsilon \downarrow 0 .$$

Moreover, for getting a o(1)-bound in (5.1.30), we must have:

$$\frac{\varepsilon}{\eta^3(\varepsilon)} = o(1) , \quad \varepsilon \downarrow 0 .$$

So, in this way, one only finds o(1)-bounds on $\frac{1}{\delta(\varepsilon)}$-timescales with $\varepsilon^{1/3}/\delta(\varepsilon) = o(1)$.

For $\delta(\varepsilon) = \varepsilon^p$, $0 \leq p < \frac{1}{3}$ we get:

$$(5.1.31)\quad \frac{\varepsilon}{\delta(\varepsilon)}\left(\Big|\int_{-\infty}^{\infty} \mathrm{II}\,dk\Big| + \Big|\frac{\partial}{\partial\bar{x}}\int_{-\infty}^{\infty} \mathrm{II}\,dk\Big|\right) \leq C\varepsilon^{1-3p}\, e^{-\alpha\varepsilon^p(\bar{x}+y)} ,$$

with α an arbitrary positive constant.
If we take $N = 1$, so $f(u) = \tilde{f}(x-\varphi_s)$, the results will improve.
Taking $\varphi(\tau,\varepsilon) = \varphi_s(\tau,\varepsilon)$, we get:

$$(5.1.32)\quad \frac{\varepsilon}{\delta}\left(\Big|\int_{-\infty}^{\infty} \mathrm{II}\,dk\Big| + \Big|\frac{\partial}{\partial\bar{x}}\int_{-\infty}^{\infty} \mathrm{II}\,dk\Big|\right) \leq C\,\frac{\varepsilon}{\eta^3}\, e^{-2\eta(\bar{x}+y)}\, e^{8\eta^3\frac{\tau}{\delta}} .$$

Now, sufficient conditions for getting an o(1)-bound are:

(5.1.33) $\frac{\eta^3(\varepsilon)}{\delta(\varepsilon)} = O(1)$; $\frac{\varepsilon}{\eta^3(\varepsilon)} = o(1)$.

(5.1.33) can be satisfied on all $1/\delta(\varepsilon)$-timescales, with $\varepsilon/\delta(\varepsilon) = o(1)$.

For $\delta(\varepsilon) = \varepsilon^p$, $0 \leqq p < 1$, we get:

(5.1.34) $$\frac{\varepsilon}{\delta}\left(\left|\int_{-\infty}^{\infty} \mathrm{II}\,dk\right| + \left|\frac{\partial}{\partial \bar{x}}\int_{-\infty}^{\infty} \mathrm{II}\,dk\right|\right) \leqq C\varepsilon^{1-p}\, e^{\beta\tau}\, e^{-\alpha\varepsilon^q(\bar{x}+y)} ,$$

with α, β positive constants and $p \leqq 3q < 1$.

For solutions of the (p)KdV for which (5.1.29) holds, we can summarize the results on condition (4.1.55c) as follows:

Define: $\bar{x} = x - \varphi(\tau,\varepsilon)$, *with* $\varphi(\tau,\varepsilon) \geqq v\,\frac{\tau}{\delta(\varepsilon)}$ *for some* $v > 0$; $\delta(\varepsilon) = \varepsilon^p$, $0 \leqq p \leqq 1$.
We have:

$$|\Omega_c(x+y,\tau)| + \left|\frac{d}{dx}\Omega_c(x+y,\tau)\right| \leqq H_1(x+y,\tau) + H_2(x+y,\tau)$$

with

1°. *If* (4.1.48) *is satisfied, then*

$$H_1(x+y,\tau) \leqq Ce^{-2\mu(\bar{x}+y)}\, e^{-2\mu(v-4\mu^2)\frac{\tau}{\delta(\varepsilon)}} , \quad \bar{x} \geqq -M(\tau,\varepsilon) ,\ y \geqq 0 ,$$
$$\tau \in [m\delta(\varepsilon),A] ,\ 0 \leqq p \leqq 1 .$$

Here μ *is a constant with* $0 < \mu \leqq \eta$ *and* $v - 4\mu^2 > 0$.

2°. *If* (5.1.25) *is satisfied and if* $\eta(\varepsilon)$ *exists, with:*

a) $0 < \eta(\varepsilon) < M_1 \leqq k_1(\tau)$;

b) $\int_{-\infty}^{\infty} f(u(x,\tau))e^{2\eta(\varepsilon)x}\,dx$ *converges;*

c) $\mathrm{II}(k,\tau)$ *is analytic on* $0 \leqq \mathrm{Im}\,k \leqq \eta(\varepsilon)$ *if* $W(0,\tau) \neq 0$.

$\mathrm{II}(k,\tau)$ *is analytic on* $\{0 \leqq \mathrm{Im}\,k \leqq \eta(\varepsilon) \cap |k| \geqq r(\varepsilon)\} \cup$
$\cup\, B_{r(\varepsilon)}(0) \setminus \{0\}$ *if* $W(0,\tau) = 0$.

(Of course, it is also sufficient if the second condition holds for all $\tau \in [0,A]$*.)*,

then:

For $N > 1$ *and* $\varphi(\tau,\varepsilon) \leqq \varphi_1(\tau,\varepsilon)$, *we have:*

$$H_2(x+y,\tau) \leqq C\varepsilon^{1-3p} e^{-\alpha\varepsilon^p(\bar{x}+y)} \quad , \quad 0 \leqq p < \frac{1}{3}, \quad \tau \in [0,A], \quad \bar{x} \geqq -M(\tau,\varepsilon), \ y \geqq 0 .$$

For $N = 1$ *and* $\varphi(\tau,\varepsilon) = \varphi_s(\tau,\varepsilon)$, *we have:*

$$H_2(x+y,\tau) \leqq C\varepsilon^{1-p} e^{-\alpha\varepsilon^q(\bar{x}+y)} \quad , \quad 0 \leqq p \leqq 3q < 1 .$$

For $\sigma(x,\tau)$ in Theorem (4.1.3) we have:

(5.1.35) $$\sigma(x,\tau) = C\left(e^{-2\mu\bar{x}} e^{-2\mu(v-4\mu^2)\frac{\tau}{\delta(\varepsilon)}} + e^{1-3p} e^{-\alpha\varepsilon^p\bar{x}}\right) = $$

$$= O\left(\varepsilon^{2\mu m(v-4\mu^2)} + \varepsilon^{1-3p}\right), \ \textit{uniformly on } \bar{x} = x - v\frac{\tau}{\delta(\varepsilon)} \geqq -M,$$

$\tau \in [m\delta(\varepsilon)\log\frac{1}{\varepsilon}, A]$, *for* $N \geqq 1$,

with $\delta(\varepsilon) = \varepsilon^p$, $0 \leqq p < \frac{1}{3}$ *and* m, M *arbitrary constants.*

(5.1.36) $$\sigma(x,\tau) = O\left(\varepsilon^{2\mu m(v-4\mu^2)} + \varepsilon^{1-p}\right) \ \textit{uniformly on } \bar{x} = x - \varphi_1(\tau,\varepsilon) \geqq -M,$$

$\tau \in [m\delta(\varepsilon)\log\frac{1}{\varepsilon}, A]$, *for* $N = 1$,

with $\delta(\varepsilon) = \varepsilon^p$, $0 \leqq p < 1$ *and* m, M *arbitrary constants.*

It is easily seen that, on the (x,τ)-regions used in (5.1.35,36), we are in the situation described in Remark (4.1.3).
With Theorem (4.1.3), we now find:

(5.1.37) $$|u(x,\tau) - u_s(x,\tau)| \leqq \sigma(x,\tau) \ , \quad \bar{x} \geqq -M, \quad \tau \in [m\delta(\varepsilon)\log\frac{1}{\varepsilon}, A]$$

with m, M *arbitrary constants and:*

a) $\bar{x} = x - v\frac{\tau}{\delta(\varepsilon)}$, $\delta(\varepsilon) = \varepsilon^p$, $0 \leqq p < \frac{1}{3}$ *and* $\sigma(x,\tau)$ *given by* (5.1.35) *if* $N \geqq 1$;

b) $\bar{x} = x - \varphi_1(\tau,\varepsilon)$, $\delta(\varepsilon) = \varepsilon^p$, $0 \leq p < 1$ *and* $\sigma(x,\tau)$ *given by* (5.1.36) *if* $N = 1$.

No results are thus obtained that are useful for solutions with an N-soliton structure on the $1/\varepsilon$-timescale.

In the next section we present a consistent perturbation theory on the $1/\varepsilon$-timescale.

V.2. Consistency results on the $1/\varepsilon$-timescale

In the introduction we gave a brief explanation of the way in which we want to obtain our asymptotic results. We will again outline the method and indicate which of the necessary steps have been derived so far.

In the first step we gave asymptotic results for 'the soliton part' $u_s(x,t)$ of a solution $u(x,t)$ of the pKdV, as well as for the eigenfunctions $\psi_{ns}(x,t)$ of $u_s(x,t)$, without specifying the exact behaviour of the spectral data. This has been done in Chapter III.

The second step consists of giving asymptotic results for $u - u_s$ and $\psi_n - \psi_{ns}$, again without specifying the exact behaviour of the spectral data. Theorems that provide results in that direction are given in Chapter IV, in particular Theorem (4.1.3) and Theorem (4.2.2). The conditions of these theorems require certain behaviour of Ω_c, respectively r. In case of the pKdV, using (3.1.5), (4.1.6), (2.2.55) and (4.2.2), these conditions can be considered as conditions on u and its (generalized) eigenfunctions. We are not able to verify these conditions for solutions of the pKdV on the $1/\varepsilon$-timescale. However, we can prove consistency of the approximations, by demonstrating that the conditions are satisfied by u_s and its (generalized) eigenfunctions. This will be done in the sequel of this section.

We need the results of the first and the second step to give an approximation for the eigenvalues. That is, we will show that using the consistency results, we can approximate $\int_{-\infty}^{\infty} f(u)\psi_n^2\,dx$ by $\int_{-\infty}^{\infty} f(u_s)\psi_{ns}^2\,dx$.

On its turn, $k_n^{-1} \cdot {}_{-\infty}\int^{\infty} f(u_s)\psi_{ns}^2\,dx$ can be approximated by some function g depending on k_n only, using the results of step 1.
This finally leads to the result that $k_n(\tau)$, being the solution of:

$$\begin{cases} \dfrac{dk_n}{d\tau} = -\dfrac{\varepsilon}{2\delta(\varepsilon)k_n} \displaystyle\int_{-\infty}^{\infty} f(u)\psi_n^2\,dx\ , \\ \\ k_n(0) = \kappa_n\ , \end{cases}$$

can be approximated by the solution $k_n^o(\tau)$ of the ordinary differential equation:

$$\begin{cases} \dfrac{dk_n^o}{d\tau} = -\dfrac{\varepsilon}{2\delta(\varepsilon)}\, g(k_n^o)\ , \\ \\ k_n^o(0) = \kappa_n\ , \end{cases}$$

where

$$g(k_n) := \tfrac{1}{2} \int_{-\infty}^{\infty} f(-2k_n^2 \operatorname{sech}^2 k_n x)\operatorname{sech}^2 k_n x\ .$$

Only in proving that ${}_{-\infty}\int^{\infty} f(u)\psi_n^2\,dx$ can be approximated by ${}_{-\infty}\int^{\infty} f(u_s)\psi_{ns}^2\,dx$, we need the results of the second step. The other parts of step 3 give no difficulties (as we will see in the next chapter).

As explained, consistency is obtained if the conditions of Theorems (4.1.3) and (4.2.2) are satisfied, when in these conditions we replace the occurring quantities u, ψ_n, etc., by u_s, ψ_{ns}, etc. To verify this, we use the expressions (2.2.53) and (2.2.54) for the generalized eigenfunctions $\psi(x,k)$.
In particular, for the pure one-soliton potential, we have:

$$u(x,t) = -2k_n^2 \operatorname{sech}^2 k_n(x-p_n) = -4k_n\psi_n^2(x,t) \Rightarrow$$

$$\Rightarrow \psi_n^2(x,t) = \tfrac{1}{2}k_n \operatorname{sech}^2 k_n(x-p_n) \Rightarrow$$

$$\Rightarrow 0 < c_n(t) = \lim_{x\to\infty} \psi_n(x,t)e^{k_n x} = \sqrt{2k_n}\, e^{k_n p_n} \Rightarrow$$

$$\Rightarrow c_n(t)\psi_n(x,t) = k_n e^{k_n p_n} \operatorname{sech} k_n(x-p_n) ,$$

which leads to:

$$(5.2.1) \qquad \psi(x,k,t) = e^{-ikx}\left\{1 - \frac{k_n(t)e^{-k_n(t)(x-p_n(t))}}{k_n(t)+ik} \operatorname{sech} k_n(t)(x-p_n(t))\right\} .$$

We define for $k \in \mathbb{R}$:

$$(5.2.2) \qquad \psi_s(x,k,t) = e^{-ikx}\left\{1 - \sum_{n=1}^{N} \frac{c_n(t)\psi_{ns}(x,t)e^{-k_n(t)x}}{k_n(t)+ik}\right\} ;$$

$$(5.2.3) \qquad b_s(k,t) = \begin{cases} \dfrac{\varepsilon}{2ik} \displaystyle\int_0^t e^{8ik^3(t-t')}\left(\int_{-\infty}^{\infty} f(u_s(x,t'))\psi_s^2(x,k,t')dx\right)dt' , \\ \qquad\qquad\qquad \text{for } |k| \geq \varepsilon\delta^{-\frac{1}{2}}(\varepsilon) \\ \tilde{b}(k,t) , \quad \text{for } |k| < \varepsilon\delta^{-\frac{1}{2}}(\varepsilon) \quad \text{(see (5.1.5))}; \end{cases}$$

$$(5.2.4) \qquad \Omega_{cs}(\xi,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{2ik\xi} b_s(k,t)dk ;$$

$$(5.2.5) \qquad r_s(x,t) = -\frac{2i}{\pi} \int_{-\infty}^{\infty} kb_s(-k,t)\psi_s^2(x,k,t)dk .$$

Note that when deriving results with $\psi_s(x,k,t)$ instead of $\psi(x,k,t)$ in the expression $\int_{-\infty}^{\infty} f(u)\psi^2 dx$, we can only speak of consistency when we take $U(x)$ to be reflectionless.

In view of the theory presented in the previous section, however, the condition of starting reflectionless seems to be more or less artificial. There, when deriving approximations for $u-u_s$ and $\psi_n - \psi_{ns}$ by means of contour integration in the complex k-plane, we have separated Ω_c into two parts. The first part depending explicitly on $b(k,0)$, and the second on the reflection generated by the perturbation. From Lemma (5.1.1), we know that the first part gives no problems when deriving bounds on Ω_c. However, one should realize that, though in an implicit way, also the second part depends on $b(k,0)$.

We have the following theorem:

Theorem (5.2.1):

Let the perturbation f *be of the following form:*

(5.2.6) a) $$f(u) = \left(\sum_{\ell=0}^{q} a_\ell \prod_{s=0}^{j_\ell} \left(\frac{\partial^s u}{\partial x^s}\right)^{p_{s\ell}}\right) L(u) , \quad a_\ell \in \mathbb{R},\ p_{s\ell} \in \mathbb{N},$$

and $L \in C^m(\mathbb{R})$ *is a function of the real variable* u *satisfying:*

b) $|L^{(m)}(u) - L^{(m)}(v)| \leq C|u-v|$ *uniformly on compacta* $K \subset \mathbb{R}$.

c) $\int_{-\infty}^{\infty} f\left(-2k_n^2(t)\ \mathrm{sech}^2\, k_n(t)x\right) e^{-2ikx}\, dx$ *is differentiable to* k_n

with a uniformly bounded derivative on $\tau \in [0,A]$, $k \in \mathbb{R}$.

Then

(5.2.7) $$2ikb_s(k,\tau) = \begin{cases} O\left(\varepsilon + \frac{1}{|k|}\varepsilon^2\delta^{-1}(\varepsilon)\tau\right), & \varepsilon\delta^{-\frac{1}{2}}(\varepsilon) \leq |k| \leq 1, \\ & \tau \in [0,A] \\ O\left(\frac{\varepsilon}{|k|^m}\right), & k \in \mathbb{R},\ |k| \geq 1,\ \textit{uniformly on } \tau \in [0,A]. \end{cases}$$

Proof:

The proof is based on the results obtained from § III.2. We will start with giving a review of the notations used and the results required.

Notations:

$$\varphi_n(t) = \frac{4}{k_n(t)} \int_0^t k_n^3(t')dt' + \frac{\varepsilon}{k_n(t)} \int_0^t \frac{H_n(t')}{2k_n(t')}\, dt' \ ;$$

$$\delta_n^+(t) = \frac{1}{2k_n(t)} \log\left(\frac{c_n^2(0)}{2k_n(t)} \prod_{i=n+1}^{N} \left(\frac{k_n(t) - k_i(t)}{k_n(t) + k_i(t)}\right)^2\right) ;$$

(5.2.8) $$p_n(t) = \varphi_n(t) + \delta_n^+(t) \ ;$$

$$z_n = x - \varphi_n(t) \ ; \quad \tilde{z}_n = x - p_n(t) \ ;$$

$$E_n(t) = \{x \in \mathbb{R} \mid \tfrac{1}{2}(\varphi_{n-1}(t) + \varphi_n(t)) \leq x \leq \tfrac{1}{2}(\varphi_{n+1}(t) + \varphi_n(t))\},$$

$n = 2,\ldots,N-1$;

$$E_1(t) = \left(-\infty, \tfrac{1}{2}(\varphi_1(t)+\varphi_2(t))\right) \;,\quad E_N(t) = \left(\tfrac{1}{2}(\varphi_N(t)+\varphi_{N-1}(t)), \infty\right) \;;$$

$$E_n^c(t) = \mathbb{R} \setminus E_n(t) \;;$$

$$h_n(x,t) = c_n(t)\psi_{ns}(x,t)e^{-k_n(t)x} \;.$$

Results (Valid for $\tau \in [0,A]$ unless mentioned otherwise.)

(1) $$z_m \leqq -\sigma t \;,\quad x \in E_n(t) \;,\quad m \geqq n+1 \;,$$
$$z_m \geqq \sigma t \;,\quad x \in E_n(t) \;,\quad m \leqq n-1 \;;$$

(2) $$\left|\frac{\partial^k}{\partial x^k} \psi_{ms}^2(x,t)\right| \leqq C\left(e^{-k_m z_m} + e^{k_m z_m}\right)^{-2} \;,\quad x \in \mathbb{R} \;;$$

(3) $$\left|\frac{\partial^k}{\partial x^k} h_m(x,t)\right| \leqq \frac{C}{1 + e^{2k_m z_m}} \leqq C \;,\quad x \in \mathbb{R} \;;$$

(4) $$\left|\frac{\partial^k}{\partial x^k} h_m(x,t)\right| \leqq \frac{C}{1 + e^{2k_m(\frac{1}{2}\varphi_{n-1}+\frac{1}{2}\varphi_n-\varphi_m)}} \leqq Ce^{-\alpha t} \;,\quad x \in E_n(t) \;,\; m \leqq n-1 \;;$$

(5) $$\frac{\partial^k}{\partial x^k}\left(\psi_{ms}^2(x,t) - \tfrac{1}{2}\operatorname{sech}^2 k_m\tilde{z}_m\right) = O(e^{-\alpha t})\,\frac{\partial^k}{\partial x^k}\operatorname{sech}^2 k_m\tilde{z}_m \;,\quad x \in E_m(t) \;;$$

(6) $$\frac{\partial^j}{\partial x^j} h_m(x,t) = h_{jm}(k_n(t),\ldots,k_N(t),\tilde{z}_n)\cdot(1 + O(e^{-\alpha t})) \;,\quad x \in E_n(t) \;,\; m \geqq n \;;$$

(7a) $$\frac{\partial^k}{\partial x^k}\operatorname{sech}^2 k_m\tilde{z}_m = O(e^{-\alpha t}) \;,\quad x \in E_m^c(t) \;;$$

(7b) $$\int_{E_m^c(t)} \left|\frac{\partial^k}{\partial x^k}\operatorname{sech}^2 k_m\tilde{z}_m\right| d\tilde{z}_m = O(e^{-\alpha t}) \;.$$

We have the following trivial corollaries of these results:

(1) + (2) $\Rightarrow$

(8) $$\int_{x \in E_m(t)} \left|\frac{\partial^k}{\partial x^k}\psi_{ns}^2(x,t)\right| dx = O(e^{-\alpha t}) \;,\quad n \neq m \;.$$

(1) + (2) + (5.2.6) ⇒

(9) $$\frac{\partial^j}{\partial x^j} f(u_s(x,t)) = \frac{\partial^j}{\partial x^j} f\left(-4\sum_{n=1}^{N} k_n\psi_{ns}^2\right) =$$

$$= \frac{\partial^j}{\partial x^j} f(-4k_m\psi_{ms}^2) + \hat{g}_{mj}(x,t) \quad , \quad x \in E_m(t)$$

with

$$\hat{g}_{mj}(x,t) = O(e^{-\alpha t}) \quad , \quad x \in E_m(t) \, , \quad \text{and} \quad \int_{E_m(t)} |\hat{g}_{mj}(x,t)|\, dx = O(e^{-\alpha t}) \, .$$

(5) + (5.2.6) ⇒

(10) $$\frac{\partial^j}{\partial x^j} f(-4k_m\psi_{ms}^2) = \frac{\partial^j}{\partial x^j} f(-2k_m^2 \operatorname{sech}^2 k_m\tilde{z}_m) +$$

$$+ O(e^{-\alpha t})\, f_{0j}(-2k_m^2 \operatorname{sech}^2 k_m\tilde{z}_m) \quad , \quad x \in E_m(t) \, .$$

Here, f_{0j} is an operator with the same structure as f.

Now, we will start with the actual proof of the theorem. We have:

(5.2.9) $$2ik\, b_s(k,t) = \varepsilon \int_0^t e^{8ik^3(t-t')} \left(\int_{-\infty}^{\infty} f(u_s(x,t'))\psi_s^2(x,k,t')dx \right) dt' \quad ;$$

(5.2.10) $$\psi_s^2(x,k,t) = e^{-2ikx} - 2e^{-ikx} \sum_{n=1}^{N} \frac{h_n(x,t)}{k_n + ik} + e^{-2ikx}\left(\sum_{n=1}^{N} \frac{h_n(x,t)}{k_n + ik}\right)^2 =$$

$$=: e^{-2ikx} + H(x,k,t) \; .$$

We define

(5.2.11) a) $f_m(k_m,\tilde{z}_m) = f(-2k_m^2 \operatorname{sech}^2 k_m\tilde{z}_m)$;

b) $\chi_n(k,k_n(t)) = \int_{\mathbb{R}} f_n(k_n,\tilde{z}_n) e^{-2ik\tilde{z}_n} d\tilde{z}_n$;

c) $g_m(x,t) = f(u_s(x,t)) - f_m(k_m,\tilde{z}_m)$.

Using (9,10), respectively, (8) and (5.2.6), we see that:

$$(5.2.12) \qquad \int_{E_m(t)} \left|\frac{\partial^j}{\partial x^j} g_m(x,t)\right| dx \leqq Ce^{-\alpha t} ;$$

$$(5.2.13) \qquad \int_{E_m^c(t)} \left|\frac{\partial^j}{\partial x^j} f_m(k_m,\tilde{z}_m)\right| d\tilde{z}_m \leqq Ce^{-\alpha t} .$$

At first, we consider the contribution in the integral (5.2.9) coming from the e^{-2ikx}-part of $\psi_s^2(x,k,t)$:

$$(5.2.14) \qquad \int_{-\infty}^{\infty} f(u_s(x,t))e^{-2ikx}\,dx = \sum_{n=1}^{N} \int_{E_n(t)} f(u_s(x,t))e^{-2ikx}\,dx =$$

$$= \sum_{n=1}^{N} \left\{ e^{-2ikp_n(t)} \int_{E_n(t)} f_n(k_n,\tilde{z}_n)e^{-2ik\tilde{z}_n} dz_n + \int_{E_n(t)} g_n(x,t)e^{-2ikx}\,dx \right\} =$$

$$= \sum_{n=1}^{N} \left\{ e^{-2ikp_n(t)} \cdot \chi_n(k,k_n(t)) - e^{2ikp_n(t)} \cdot \int_{E_n^c(t)} f_n(k_n,\tilde{z}_n)e^{-2ik\tilde{z}_n} d\tilde{z}_n + \right.$$

$$\left. + \int_{E_n(t)} g_n(x,t)e^{-2ikx}\,dx \right\} .$$

From (5.2.12), respectively (5.2.13), we see that:

$$(5.2.15) \qquad \varepsilon \int_0^t \int_{E_n(t')} |g_n(x,t')|\,dx\,dt' = O(\varepsilon) ;$$

$$\varepsilon \int_0^t \int_{E_n^c(t')} |f_n(k_n,\tilde{z}_n)|\,d\tilde{z}_n\,dt' = O(\varepsilon) .$$

Using (5.2.6c) and

$$\frac{dk_n}{dt} = O(\varepsilon) , \quad \frac{dp_n}{dt} = 4k_n^2(t) + O(\varepsilon) , \quad \frac{d^2}{dt^2} p_n = O(\varepsilon) ,$$

we find:

$$(5.2.16)\quad \left|e^{8ik^3t}\int_0^t e^{-2ik(4k^2t'+p_n(t'))}\cdot\chi_n(k,k_n(t'))dt'\right| =$$

$$= \left|e^{8ik^3t}\int_0^t e^{-2ik(4k^2t'+p_n(t'))}(4k^2+\dot{p}_n(t'))(4k^2+\dot{p}_n(t'))^{-1}\cdot\right.$$

$$\left.\cdot\ \chi_n(k,k_n(t'))dt'\right| =$$

$$= \left|e^{8ik^3t}\left[-\frac{1}{2ik}e^{-2ik(4k^2t'+p_n(t'))}(4k^2+\dot{p}_n(t'))^{-1}\cdot\chi_n(k,k_n(t'))\right]_0^t + \right.$$

$$+\frac{e^{8ik^3t}}{2ik}\int_0^t e^{-2ik(4k^2t'+p_n(t'))}\cdot$$

$$\left.\cdot\left(\frac{-1}{(4k^2+\dot{p}_n)^2}\ddot{p}_n\chi_n + \frac{1}{4k^2+\dot{p}_n}\frac{\partial\chi_n}{\partial k_n}\frac{dk_n}{dt'}\right)dt'\right| =$$

$$= O\left(\frac{1+\varepsilon t}{k(1+k^2)}\right).$$

Combining (5.2.14,15,16) gives:

$$(5.2.17)\quad \varepsilon\int_0^t e^{8ik^3(t'-t)}\left(\int_{-\infty}^{\infty} f(u_s(x,t'))e^{-2ikx}dx\right)dt' =$$

$$= O\left(\varepsilon\left(1+\frac{\varepsilon t}{k(1+k^2)}\right)\right) = O\left(\varepsilon\left(1+\frac{\varepsilon\delta^{-1}(\varepsilon)\tau}{k(1+k^2)}\right)\right),\quad \tau\in[0,A].$$

Now, we consider the contribution in the integral (5.2.9), coming from the $H(x,k,t)$-part of $\psi_s^2(x,k,t)$:

$$(5.2.18)\quad \int_{-\infty}^{\infty} f(u_s(x,t))H(x,k,t)dx = \sum_{n=1}^{N}\int_{E_n(t)} f(u_s(x,t))H(x,k,t)dx =$$

$$= \sum_{n=1}^{N}\int_{E_n(t)} f(u_s(x,t))\cdot$$

$$\cdot\left(-2e^{-ikx}\sum_{m=1}^{n-1}\frac{h_m(x,t)}{k_m+ik}+e^{-2ikx}\sum_{\substack{m\leq n-1\\ \text{or}\\ \ell\leq n-1}}\frac{h_m(x,t)h_\ell(x,t)}{(k_m+ik)(k_\ell+ik)}\right)dx\ +$$

$$+\sum_{n=1}^{N}\int_{E_n(t)} f(u_s(x,t))\ \cdot$$

$$\cdot\left(-2e^{-ikx}\sum_{m=n}^{N}\frac{h_m(x,t)}{k_m+ik}+e^{-2ikx}\sum_{m,\ell\geq n}\frac{h_m(x,t)h_\ell(x,t)}{(k_m+ik)(k_\ell+ik)}\right)dx\ .$$

For the first $\sum_{n=1}^{N}$-summation in (5.2.18) it is easy to find a bound, since with (4) it follows that:

$$(5.2.19)\qquad \int_{E_n(t)} |f(u_s(x,t))\left(-2e^{-ikx}\sum_{m=1}^{n-1}\ldots+e^{-2ikx}\sum_{\substack{m\leq n-1\\ \text{or}\\ \ell\leq n-1}}\ldots\right)|\,dx\leq Ce^{-\alpha t}\ .$$

The second part can be estimated as follows:

$$(5.2.20)\qquad \int_{E_n(t)} f(u_s(x,t))e^{-ikx}\frac{h_m(x,t)}{k_m+ik}\,dx=$$

$$=\left(\int_{E_n(t)} f_n(k_n,\tilde z_n)e^{-ik\tilde z_n}\frac{h_m(x,t)}{k_m+ik}\,dx\right)e^{-ikp_n(t)}\ +$$

$$+\int_{E_n(t)} g_n(x,t)e^{-ikx}\frac{h_m(x,t)}{k_m+ik}\,dx\ ,\quad m\geq n\ .$$

Using (6), (5.2.12) and (5.2.20) we get

$$|\int_0^t e^{8ik^3(t-t')}\int_{E_n(t')} f(u_s(x,t'))e^{-ikx}\frac{h_m(x,t')}{k_m+ik}\,dx|\leq$$

$$\leq C+|e^{8ik^3t}\int_0^t e^{-ik(8k^2t'+p_n(t'))}\ \cdot$$

$$\cdot\left\{\int_{E_n(t')}\frac{e^{-ik\tilde z_n}}{k_m+ik}f_n(k_n,\tilde z_n)h_{0m}(k_n,\ldots,k_N,\tilde z_n)d\tilde z_n\right\}dt'|\ ,\quad m\geq n\ .$$

Definition:

(5.2.21) a) Since $(\partial^j/\partial x^j)h_m(x,\tau)$ is uniformly bounded on $\mathbb{R}\times[0,A]$, we can easily extend the function $h_{jm}(x,\tau)$ defined for $x \in E_n(\tau)$, $m \geqq n$, to a function $h^u_{jm}(x,\tau)$ which is uniformly bounded in (x,τ) on $\mathbb{R}\times[0,A]$.

b) $$\chi_{nm}(k,k_n,\dots,k_N) = \int_{\mathbb{R}} \frac{f_n(k_n,\tilde{z}_n)}{k_m + ik}\, e^{-ik\tilde{z}_n}\, h^u_{0m}(k_n,\dots,k_N,\tilde{z}_n)\,d\tilde{z}_n \,.$$

Completely analogous to (5.2.15,16), we now find that:

(5.2.22) a) $$\varepsilon\int_0^t \int_{E_n^c(t')} \left|\frac{f_n(k_n,\tilde{z}_n)}{k_m + ik}\, h^u_{0m}(k_n,\dots,k_N,\tilde{z}_n)\right| d\tilde{z}_n\,dt' = O(\varepsilon) \;;$$

b) $$\left|e^{8ik^3t}\int_0^t e^{-ik(8k^2t'+p_n(t'))}\cdot\chi_{nm}(k,k_n(t'),\dots,k_N(t'))dt'\right| =$$

$$= O\left(\frac{1+\varepsilon t}{k(1+k^2)}\right) .$$

It is obvious that the same method is suitable for the part:

$$\int_{E_n(t)} f(u_s(x,t))e^{-ikx}\,\frac{h_m(x,t)h_\ell(x,t)}{(k_m+ik)(k_\ell+ik)}\,dx \;,\quad m,\ell \geqq n \,.$$

So, combining (5.2.18) to (5.2.22), we find:

(5.2.23) $$\varepsilon\int_0^t e^{8ik^3(t-t')}\left(\int_{-\infty}^{\infty} f(u_s(x,t'))H(x,k,t')dx\right)dt' =$$

$$= O\left(\varepsilon\left(1+\frac{\varepsilon t}{k(1+k^2)}\right)\right) = O\left(\varepsilon\left(1+\frac{\varepsilon\delta^{-1}(\varepsilon)}{k(1+k^2)}\right)\right) \quad \text{uniformly on } \tau \in [0,A] \,.$$

Finally, with (5.2.9,17,23) we find

(5.2.24) $$kb_s(k,t) = \begin{cases} O\left(\varepsilon + \frac{\varepsilon^2 t}{|k|}\right) = O\left(\varepsilon + \frac{\varepsilon^2\delta^{-1}(\varepsilon)\tau}{|k|}\right), & \tau \in [0,A]\,,\; |k| \leqq 1 \,,\\[2ex] O(\varepsilon) \text{ uniformly on } \tau \in [0,A]\,, & \text{for } |k| \geqq 1 \,. \end{cases}$$

We can improve the estimate for $|k| \geq 1$ in the following way:
Since u_s and its x-derivatives tend to zero for $|x| \to \infty$, by m times partial integration in $\int_{-\infty}^{\infty} f(u_s)e^{-2ikx}dx$ we find:

$$\int_{-\infty}^{\infty} f(u_s)e^{-2ikx}\,dx = \left(\frac{1}{2ik}\right)^m \int_{-\infty}^{\infty} e^{-2ikx}\,\frac{\partial^m}{\partial x^m} f(u_s)dx\ .$$

Analogously:

$$\int_{-\infty}^{\infty} f(u_s)\left(-2e^{-ikx}\sum_{n=1}^{N}\frac{h_n}{k_n+ik} + e^{-2ikx}\left(\sum_{n=1}^{N}\frac{h_n}{k_n+ik}\right)^2\right)dx =$$

$$= -2\left(\frac{1}{ik}\right)^m \int_{-\infty}^{\infty} e^{-ikx}\,\frac{\partial^m}{\partial x^m}\left(f(u_s)\sum_{n=1}^{N}\frac{h_n}{k_n+ik}\right)dx\ +$$

$$+ \left(\frac{1}{2ik}\right)^m \int_{-\infty}^{\infty} e^{-2ikx}\,\frac{\partial^m}{\partial x^m}\left(f(u_s)\left(\sum_{n=1}^{N}\frac{h_n}{k_n+ik}\right)^2\right)dx\ .$$

For the x-derivatives of ψ_{ns} and h_n, we have the same bounds as for ψ_{ns} and h_n. Moreover, also (6), (10) hold for all $j \in \mathbb{N}$. Therefore, by working completely in the same way as is used to derive (5.2.24), we find that for $m \in \mathbb{N}$ for which (5.2.6) holds, the bounds in (5.2.24) can be multiplied by a factor k^{-m}, without losing their validity. This leads to (5.2.7).

Q.E.D.

Remark (5.2.1):
We have proved the theorem for perturbations of the form (5.2.6), because these are the perturbations to which we have to restrict ourselves later. (See Chapter VI.) From the proof, however, it is evident that the theorem holds for a larger class of perturbations. It is sufficient to assume that the perturbation $f = f(u,u^{(1)},\ldots,u^{(p)})$ is a function of u and x-derivatives of u up to a certain power p, that satisfies:

i) $f \in C^m(\mathbb{R}^{p+1} \to \mathbb{R})$;

ii) $|f^{(m)}(x_0,\ldots,x_p) - f^{(m)}(y_0,\ldots,y_p)| \leq C\|\vec{x}-\vec{y}\|$,
uniformly on compacta $K \subset \mathbb{R}^{p+1}$;

iii) Condition (5.2.6c).

Corollaries of Theorem (5.2.1):

1) Using

$$|b_s(k,\tau)| = |\tilde{b}(k,\tau)| \leq |b(k,\tau)| + |b(k,0)| \leq 2$$

for $|k| < a(\varepsilon) := \varepsilon\delta^{-1}(\varepsilon)$ and (5.2.7) we get

(5.2.25) $kb_s(k,\tau) = O(\varepsilon\delta^{-\frac{1}{2}}(\varepsilon))$, *uniformly in* (k,τ) *on* $\{|k| \leq 1\} \times [0,A]$.

2) Let (5.2.6) hold for m = 2, then:

$$|r_s(x,\tau)| \leq \frac{2}{\pi} \int_{-\infty}^{\infty} |kb_s(-k,\tau)||\psi_s^2(x,k,\tau)|dk \leq$$

$$\leq C\left\{ \int_{-\infty}^{-1} |kb_s(-k,\tau)|dk + \int_{-1}^{-a(\varepsilon)} |kb_s(-k,\tau)|dk + \int_{-a(\varepsilon)}^{a(\varepsilon)} |kb_s(-k,\tau)|dk + \right.$$

$$\left. + \int_{a(\varepsilon)}^{1} |kb_s(-k,\tau)|dk + \int_{1}^{\infty} |kb_s(-k,\tau)|dk \right\} .$$

For ${}_{-\infty}\int^{-1}$ and ${}_{1}\int^{\infty}$ we use (5.2.7), ($|k| \geq 1$), to find that these integrals are $O(\varepsilon)$.

For ${}_{-1}\int^{-a}$ and ${}_{a}\int^{1}$ we use (5.2.7), ($|k| \leq 1$), to find that these integrals are $O(\varepsilon + \varepsilon^2\delta^{-1}(\varepsilon)\log a(\varepsilon))$.

For ${}_{-a}\int^{a}$ we use $|b_s(k,\tau)| \leq 2$, to find that this integral is $O(a^2(\varepsilon))$.

We again take: $\delta(\varepsilon) = \varepsilon^p$, $0 \leq p \leq 1$. Since $a(\varepsilon) = \varepsilon\delta^{-\frac{1}{2}}(\varepsilon)$, this leads to

(5.2.26) $r_s(x,\tau) = O(\varepsilon + \varepsilon^2\delta^{-1}(\varepsilon)\log\varepsilon)$ uniformly in (x,τ) on $\mathbb{R} \times [0,A]$.

3) Analogously we find

(5.2.27) a) $\Omega_{cs}(\xi,\tau) = O(\varepsilon\delta^{-\frac{1}{2}}(\varepsilon))$ uniformly in (ξ,τ) on $\mathbb{R} \times [0,A]$;

b) $\frac{\partial}{\partial\xi}\Omega_{cs}(\xi,\tau) = O(\varepsilon + \varepsilon^2\delta^{-1}(\varepsilon)\log\varepsilon)$

uniformly in (ξ,τ) on $\mathbb{R} \times [0,A]$.

This implies, that in any case for $b(k,0) \equiv 0$, we have satisfactorily consistency results for conditions (4.1.55c) and (4.2.62b). (And also for (4.2.57) with $k = 0$.)
Since (4.1.55d) can only give problems in the exceptional case that $\|ST_c\beta_c\| \sim \|\beta_c\|$ and $\|T\beta_c\| \sim \|\beta_c\|$, we will not go into this condition any further.

We finish this chapter by concluding that we have shown consistency of the following results:

(5.2.28) a) $|u(x,\tau) - u_s(x,\tau)| = O(q(\varepsilon))$

uniformly on $x \in I(\varepsilon,\tau)$, $\tau \in [m(\varepsilon),A]$;

b) $|\psi_n(x,\tau) - \psi_{ns}(x,\tau)| = O(q(\varepsilon))$

uniformly on $x \in I(\varepsilon,\tau)$, $\tau \in [m(\varepsilon),A]$,

with

(5.2.29) $\delta(\varepsilon) = \varepsilon^{p}$, $0 \leq p \leq 1$, $q(\varepsilon) = \varepsilon^{1-\frac{1}{2}p}$ *and*

a) $m(\varepsilon) = 0$, $I(\varepsilon,\tau) = \mathbb{R}$ *if* $b(k,0) \equiv 0$;

b) $m(\varepsilon) = m\delta(\varepsilon)\log\frac{1}{\varepsilon}$, $I(\varepsilon,\tau) = [M + v\frac{\tau}{\delta(\varepsilon)}, \infty)$

with m, v *positive constants (*m *taken so large that in the bound* (5.1.7) *we have* $\alpha m \geq 1-\frac{1}{2}p$*) and* M *an arbitrary constant, if* $b(k,0) \not\equiv 0$.
As to not to lose information about the soliton structure, v *must be such that there exists a positive constant* $\tilde{v}$, *with:*

$$v\frac{\tau}{\delta(\varepsilon)} < \tilde{v}\frac{\tau}{\delta(\varepsilon)} \leq \varphi_1(\tau,\varepsilon) .$$

In the next chapter we will give the third step of the perturbation analysis, such as outlined at the beginning of this section. We take as a starting point Theorem (3.2.1) with corollaries, and the estimates (5.2.28), with $q(\varepsilon)$ unspecified and $m(\varepsilon) = m\delta(\varepsilon)\log\frac{1}{\varepsilon}$, $I(\varepsilon,\tau) = [M + v\frac{\tau}{\delta(\varepsilon)}, \infty)$ if $b(k,0) \not\equiv 0$, respectively $m(\varepsilon) = 0$, $I(\varepsilon,\tau) = \mathbb{R}$ if $b(k,0) \equiv 0$.

CHAPTER VI

EXPLICIT APPROXIMATIONS FOR SOLUTIONS OF THE pKdV-INITIAL VALUE PROBLEM

Our final task is to derive approximations for the eigenvalues. The evolution of the eigenvalues is given by:

$$(6.1) \qquad -2k_n \frac{d}{d\tau} k_n = \frac{\varepsilon}{\delta(\varepsilon)} \int_{-\infty}^{\infty} f(u(x,\tau))\psi_n^2(x,\tau)\,dx \;.$$

In this chapter, the perturbation $f(u)$ has the following structure:

(6.2) $f(u)$ is as in (5.2.6a), with $L \in C^j(\mathbb{R})$, where $j = \max\{j_0,\ldots,j_q\}$. (If $j = 0$, then the only condition on L is that L must be Lipschitz continuous uniformly on compacta.)

We define:

$$(6.3) \qquad g(k) = \tfrac{1}{2} \int_{-\infty}^{\infty} f(-2k^2 \operatorname{sech}^2 kx) \operatorname{sech}^2 kx\,dx \;.$$

First, we show that as a consequence of Theorem (3.2.1) we have the following lemma:

Lemma (6.1):

$$\int_{-\infty}^{\infty} f(u_s(x,\tau))\psi_{ms}^2(x,\tau)\,dx - k_m\,g(k_m) = O\left(e^{-\alpha\frac{\tau}{\delta(\varepsilon)}}\right),$$

for some positive constant α, $\tau \in [0,A]$.

Proof:

The proof is given in Appendix D.

Now we need an estimate for $\int_{-\infty}^{\infty} f(u)\psi_n^2\,dx - \int_{-\infty}^{\infty} f(u_s)\psi_{ns}^2\,dx$.

We discriminate between the situations $b(k,0) \equiv 0$ and $b(k,0) \not\equiv 0$. For $b(k,0) \not\equiv 0$ we have the following lemma:

Lemma (6.2):

If

(6.4) a) $u(x,\tau) - u_s(x,\tau) = O(q(\varepsilon))$ *uniformly on* $x \geq M + v\frac{\tau}{\delta(\varepsilon)}$, $\tau \in [m\delta(\varepsilon)\log\frac{1}{\varepsilon}, A]$,

b) $\psi_n(x,\tau) - \psi_{ns}(x,\tau) = O(q(\varepsilon))$ *uniformly on* $x \geq M + v\frac{\tau}{\delta(\varepsilon)}$, $\tau \in [m\delta(\varepsilon)\log\frac{1}{\varepsilon}, A]$,

with $q(\varepsilon)$ *unspecified,* m *a positive constant and* M, v *satisfying* (5.2.29),

and if moreover

(6.5) $$\frac{\partial^s u(x,\tau)}{\partial x^s}, \quad s = 0,1,\ldots,2j \text{ is uniformly bounded on } \mathbb{R} \times [0,A],$$

then

(6.6) $$\int_{-\infty}^{\infty} f(u(x,\tau))\psi_n^2(x,\tau)dx - \int_{-\infty}^{\infty} f(u_s(x,\tau))\psi_{ns}^2(x,\tau)dx =$$

$$= O\left(q(\varepsilon) + e^{-\alpha\frac{\tau}{\delta(\varepsilon)}}\right)$$ *for some positive constant* α *and* $\tau \in [m\delta(\varepsilon)\log\frac{1}{\varepsilon}, A]$.

Proof:

The proof is given in Appendix D.

For $b(k,0) \equiv 0$, we use:

Lemma (6.3):

If

(6.7) a) $u(x,\tau) - u_s(x,\tau) = O(q(\varepsilon))$ *uniformly on* $x \in \mathbb{R}$, $\tau \in [0,A]$;

b) $\psi_n(x,\tau) - \psi_{ns}(x,\tau) = O(q(\varepsilon))$ *uniformly on* $x \in \mathbb{R}$, $\tau \in [0,A]$, *then:*

$$(6.8)\qquad \int_{-\infty}^{\infty} f(u(x,\tau))\psi_n^2(x,\tau)dx - \int_{-\infty}^{\infty} f(u_s(x,\tau))\psi_{ns}^2(x,\tau)dx = O(q(\varepsilon)) ,$$

uniformly in τ *on* $[0,A]$.

Proof:

The proof is a simplified analogue of the proof for Lemma (6.2) and is therefore omitted.

Before coming to the main theorem of this chapter, we will give one more lemma.

Lemma (6.4):

$g(k)$ *is uniformly Lipschitz-continuous in* k *on compacta* $K \subset \mathbb{R}$.

Proof:

Trivial, using the explicit structure of the perturbation f as given in (6.2).

Theorem (6.1):

Let (3.2.13) *and the conditions of Lemma* (6.2), *respectively Lemma* (6.3), *be satisfied for* $\tau \in [0,A]$. *Let* $k_n(\tau)$ *be the solution of:*

$$(6.9)\qquad \begin{cases} \dfrac{d}{d\tau} k_n(\tau) = - \dfrac{\varepsilon}{2\delta(\varepsilon)k_n(\tau)} \displaystyle\int_{-\infty}^{\infty} f(u(x,\tau))\psi_n^2(x,\tau)dx , \\ k_n(0) = \kappa_n . \end{cases}$$

Let $k_n^o(\tau)$ *be the solution of:*

$$(6.10)\qquad \begin{cases} \dfrac{d}{d\tau} k_n^o(\tau) = - \dfrac{\varepsilon}{2\delta(\varepsilon)} g(k_n^o(\tau)) , \\ k_n^o(0) = \kappa_n . \end{cases}$$

Then:

$$(6.11)\qquad \sup_{\tau\in[0,A]} |k_n(\tau) - k_n^o(\tau)| = O(\varepsilon[1 + q(\varepsilon)\delta^{-1}(\varepsilon)]) \quad \textit{if } b(k,0) \equiv 0 ;$$

$$(6.12)\qquad \sup_{\tau\in[0,A]} |k_n(\tau) - k_n^o(\tau)| = O(\varepsilon[\log\frac{1}{\varepsilon} + q(\varepsilon)\delta^{-1}(\varepsilon)]) \quad \textit{if } b(k,0) \not\equiv 0 .$$

Proof:

We use:

(6.13) $|g(x)-g(y)| \leqq L|x-y|$ uniformly on $M_1 \leqq x,y \leqq M_2$.

(M_1, M_2 as in (3.2.36).)

At first, we look on $\tau \in [0,A_0]$ with $A_0 \leqq A$ and $A_0 L < 2$.

For $b(k,0) \equiv 0$, we have:

$$\frac{1}{k_n}\int_{-\infty}^{\infty} f(u)\psi_n^2\,dx = g(k_n) + O(q(\varepsilon)) + O\left(e^{-\alpha\frac{\tau}{\delta(\varepsilon)}}\right)$$

$$\Rightarrow \quad k_n(\tau) = \kappa_n - \frac{\varepsilon}{2\delta(\varepsilon)}\int_0^{\tau} g(k_n(\tau'))d\tau' + O\left(\frac{\varepsilon q(\varepsilon)}{\delta(\varepsilon)}\right) + O(\varepsilon) =$$

$$= \kappa_n - \frac{\varepsilon}{2\delta(\varepsilon)}\int_0^{\tau} g(k_n^o(\tau'))d\tau' +$$

$$+ \frac{\varepsilon}{2\delta(\varepsilon)}\int_0^{\tau} [g(k_n^o(\tau')) - g(k_n(\tau'))]\,d\tau' + O(\varepsilon[1+q(\varepsilon)\delta^{-1}(\varepsilon)]) =$$

$$= k_n^o(\tau) + \frac{\varepsilon}{2\delta(\varepsilon)}\int_0^{\tau} [g(k_n^o(\tau')) - g(k_n(\tau'))]\,d\tau' +$$

$$+ O(\varepsilon[1+q(\varepsilon)\delta^{-1}(\varepsilon)]) \ .$$

$$\Rightarrow \quad \sup_{\tau\in[0,A_0]} |k_n(\tau) - k_n^o(\tau)| \leqq C\varepsilon(1+q(\varepsilon)\delta^{-1}(\varepsilon)) + \frac{\varepsilon L A_0}{2\delta(\varepsilon)} \cdot$$

$$\cdot \sup_{\tau\in[0,A_0]} |k_n(\tau) - k_n^o(\tau)|$$

$$\Rightarrow \quad \sup_{\tau\in[0,A_0]} |k_n(\tau) - k_n^o(\tau)| = O(\varepsilon[1+q(\varepsilon)\delta^{-1}(\varepsilon)]) \ .$$

For $b(k,0) \not\equiv 0$, we divide the τ-interval into $[0,\tau_m]$ and $[\tau_m,A_0]$. Here, $\tau_m := m\delta(\varepsilon)\log\frac{1}{\varepsilon}$.

For $\tau \in [0,\tau_m]$ we only have the rough approximation:

$$k_n(\tau) = k_n^o(\tau) + O(\varepsilon \log \frac{1}{\varepsilon}) \quad \text{uniformly on } [0,\tau_m] .$$

For $\tau \in [\tau_m, A_0]$ we get

$$\begin{cases} \frac{d}{d\tau} k_n = -\frac{\varepsilon}{2\delta(\varepsilon)} g(k_n) + O(q(\varepsilon)) + O\left(e^{-\alpha \frac{\tau}{\delta(\varepsilon)}}\right), \\ k_n(\tau_m) = k_n^o(\tau_m) + O(\varepsilon \log \frac{1}{\varepsilon}) \end{cases}$$

$$\Rightarrow \quad k_n(\tau) = k_n^o(\tau_m) - \frac{\varepsilon}{2\delta(\varepsilon)} \int_{\tau_m}^{\tau} g(k_n^o(\tau'))d\tau' +$$

$$+ \frac{\varepsilon}{2\delta(\varepsilon)} \int_{\tau_m}^{\tau} [g(k_n^o(\tau')) - g(k_n(\tau'))]d\tau' + O(\varepsilon[\log\frac{1}{\varepsilon} + q(\varepsilon)\delta^{-1}(\varepsilon)])$$

$$\Rightarrow \quad \sup_{\tau\in[0,A_0]} |k_n(\tau) - k_n^o(\tau)| = O(\varepsilon[\log\frac{1}{\varepsilon} + q(\varepsilon)\delta^{-1}(\varepsilon)]) .$$

Now we have proved (6.11) and (6.12) for $\tau \in [0,A_0]$. Taking A_0 as a new starting time, we can easily extend the validity-region to $[0,2A_0]$ by following the same procedure as in the proof for the case $b(k,0) \not\equiv 0$. Continuing in this way, we see that the validity-region can be extended to any interval [0,A] on which the conditions of the theorem hold.

Q.E.D.

Corollary (6.1):

The position of the n-th solution is given by:

$$p_n(\tau) := \varphi_n(\tau) + \delta_n^+(\tau) = \frac{1}{\delta(\varepsilon)} \int_0^{\tau} 4k_n^2(\tau')d\tau' +$$

$$+ \frac{1}{2k_n(\tau)} \log\left\{\frac{c_n^2(0)}{2k_n(\tau)} \prod_{i=n+1}^{N} \left(\frac{k_n(\tau) - k_i(\tau)}{k_n(\tau) + k_i(\tau)}\right)^2\right\} + O\left(\frac{\varepsilon\tau}{\delta(\varepsilon)}\right).$$

We define

$$(6.14) \qquad p_n^o(\tau) = \frac{1}{\delta(\varepsilon)} \int_0^{\tau} 4(k_n^o(\tau'))^2 d\tau' + \frac{1}{2k_n^o} \log\left\{\frac{c_n^2(0)}{2k_n^o(\tau)} \prod_{i=n+1}^{N} \left(\frac{k_n^o(\tau) - k_i^o(\tau)}{k_n^o(\tau) + k_i^o(\tau)}\right)^2\right\} .$$

(Note that on the $\frac{1}{\varepsilon}$-timescale we can just as well omit the second term in $p_n^o(\tau)$, because then, the $O(\frac{\varepsilon\tau}{\delta(\varepsilon)})$-term in $p_n(\tau)$ becomes of order 1.)

With (3.2.3b) and Theorem (6.1) we see that

$$(6.15)\qquad p_n(\tau) - p_n^o(\tau) = \begin{cases} O\left(\varepsilon(1+\frac{q(\varepsilon)}{\delta(\varepsilon)})(1+\frac{\tau}{\delta(\varepsilon)})\right) & \text{if } b(k,0) \equiv 0 , \\ O\left(\varepsilon(\log\frac{1}{\varepsilon}+\frac{q(\varepsilon)}{\delta(\varepsilon)})(1+\frac{\tau}{\delta(\varepsilon)})\right) & \text{if } b(k,0) \not\equiv 0 . \end{cases}$$

(Of course, it requires that $\varepsilon q(\varepsilon)\delta^{-1}(\varepsilon) = o(1)$.)

Remark (6.1):

As a consequence of (6.15) and Remark (3.2.2.1°), the results of Theorem (3.2.1) and corollaries valid for $x \in E_n(\tau)$, also hold on $E_n^o(\tau)$, where $E_n^o(\tau)$ is defined as:

$$(6.16)\qquad E_n^o(\tau) = \{x \in \mathbb{R} \mid \tfrac{1}{2}(p_{n-1}^o(\tau) - p_n^o(\tau)) \leq x - p_n^o(\tau) \leq \tfrac{1}{2}(p_{n+1}^o(\tau) - p_n^o(\tau))\} ,$$

$$n = 2,\dots,N-1 ;$$

$$E_1^o(\tau) = (-\infty, \tfrac{1}{2}(p_1^o(\tau) + p_2^o(\tau))] \; ; \quad E_N^o(\tau) = [\tfrac{1}{2}(p_N(\tau) + p_{N-1}(\tau)),\infty) .$$

We will now summarize the previous results and conditions that are needed to find an approximation for a solution $u(x,t)$ of the pKdV on ε^{-p}-timescales, $0 \leq p \leq 1$.

Summary of conditions needed for (3.2.49), (5.2.28,29) and (6.11,12):

(6.17) a) The perturbation f has the following form:

$$f(u) = \left(\sum_{\ell=0}^{q} a_\ell \prod_{s=0}^{j_\ell} \left(\frac{\partial^s u}{\partial x^s}\right)^{p_{s\ell}}\right) L(u) , \quad a_\ell \in \mathbb{R}, \quad p_{s\ell} \in \mathbb{N} ,$$

with $L(u) \in C^m(\mathbb{R})$, where: $m = \max\{2,j\}$, $j = \max\{j_0,\dots,j_q\}$.

If $m = 2$, then as an additional condition, we have: $D^2L(u)$ must be Lipschitz-continuous.

b) The number N of eigenvalues is invariant in time.

c) The eigenvalues satisfy: There exist positive constants M_1, M_2, μ_n, such that

$$0 < M_1 \leq k_1(\tau) < \dots < k_N(\tau) \leq M_2 \quad \text{and} \quad k_n(\tau) - k_{n-1}(\tau) \geq \mu_n\,, \quad n = 2,\dots,N\,.$$

d) In the case of $\delta(\varepsilon) = \varepsilon$, the τ-interval $[0,A]$ must be taken so that a positive constant σ exists with:

$$\delta(\varepsilon)(\varphi_{n+1}(\tau) - \varphi_n(\tau)) \geq \sigma\tau\,, \quad n = 0,\dots,N-1\,, \quad \tau \in [0,A]\,.$$

(Existence of such intervals has been proved.)

e) i) $u(x,\tau) \in C^{2j}(\mathbb{R})$ for all the values of the parameter $\tau \in [0,A]$.
All the x-derivatives of u up to degree 2j, must be uniformly bounded on $\mathbb{R} \times [0,A]$.

ii) $\int_{-\infty}^{\infty} |u(x,\tau)|\,dx$ and $\int_{-\infty}^{\infty} |f(u(x,\tau))|\,dx$ are uniformly bounded on $[0,A]$.

iii) $u(x,\tau)$ must satisfy a second order growth condition in x, that is: $\int_{-\infty}^{\infty} (1+x^2)u(x,t)\,dx$ converges.

f) $\exists \eta > 0$ with $b(k,0)$ is analytic on $0 < \mathrm{Im}\,k \leq \eta$, continuous on $0 \leq \mathrm{Im}\,k \leq \eta$, and moreover: $b(k,0) = o(|k|^2)$, $|k| \to \infty$, $0 \leq \mathrm{Im}\,k \leq \eta$.

We point out that our starting point is:
Given a perturbation f of type (6.17a), we define:

(6.18) H_f *is the class of solutions* $u(x,\tau)$ *of the pKdV satisfying* (6.17).

Our perturbation results are valid for solutions in H_f. The problem of showing existency of these solutions is not treated in this thesis.

<u>Results based on (3.2.49), (5.2.28,29) and (6.11,12):</u>

Let $u(x,t)$ be a solution in H_f of:

$$(6.19) \qquad \begin{cases} u_t - 6uu_x + u_{xxx} = \varepsilon f(u) \\ u(x,0) = U(x)\,. \end{cases}$$

We can find an approximation of $u(x,t)$ in the following way:

1°. Calculate the eigenvalues $\kappa_1,\dots,\kappa_N$ of $U(x)$.
See whether $b(k,0) \equiv 0$ or not.

2°. Derive the solutions of the O.D.E.'s

$$(6.20)\quad \begin{cases} \frac{d}{d\tau} k_n^o(\tau) = -\frac{\varepsilon}{4\delta(\varepsilon)} \int_{-\infty}^{\infty} f(-2(k_n^o)^2 \operatorname{sech}^2 k_n^o x) \operatorname{sech}^2 k_n^o x\, dx \ , \\ k_n^o(0) = \kappa_n \ . \end{cases}$$

We have to discriminate the situations $b(k,0) \not\equiv 0$ and $b(k,0) \equiv 0$.

Situation $b(k,0) \not\equiv 0$:
We define:

$$(6.21)\quad I(\varepsilon) = [m\delta(\varepsilon) \log\frac{1}{\varepsilon}, A] \ ,$$

$$D = \mathbb{R} \times I \ ,$$

$$D^+ = [M + v\tau\delta^{-1}(\varepsilon), \infty) \times I \ , \text{ where}$$

i) M is an arbitrary positive constant.

ii) v is a constant with $0 < v \leq 4M_1^2$. (In fact, we can take each positive v with $v\tau < \tilde{v}\tau \leq \varphi_1(\tau)$ for some positive constant $\tilde{v}$.)

iii) m is a positive constant, so large that $\alpha m \geq 1-\frac{1}{2}p$, with α as in (3.2.47,49), respectively (5.1.7). (In fact, the exponentially fast decaying bounds in (3.2.49), respectively (5.1.7), are better than a $O(\varepsilon^{1-\frac{1}{2}p})$-bound.)

We have:

$$(6.22)\quad \text{a)}\ \sup_{D^+} |u(x,\tau) - u_s(x,\tau)| = O(\varepsilon^{1-\frac{1}{2}p}) \ ;$$

$$\text{b)}\ \sup_{D} |u_s(x,\tau) + \sum_{n=1}^{N} 2k_n^2 \operatorname{sech}^2 k_n(x-p_n)| = O(\varepsilon^{1-\frac{1}{2}p}) \ ;$$

$$\text{c)}\ \sup_{E_m^o \times I} |u_s(x,\tau) + 2k_m^2 \operatorname{sech}^2 k_m(x-p_m) = O(\varepsilon^{1-\frac{1}{2}p}) \ .$$

Combining these results, we find:

$$(6.23)\qquad a)\quad \sup_{D^+} \left|u(x,\tau) + \sum_{n=1}^{N} 2k_n^2 \operatorname{sech}^2 k_n(x-p_n)\right| = O(\varepsilon^{1-\frac{1}{2}p}) \;;$$

$$b)\quad \sup_{D^+\cap\{E_m^o\times I\}} \left|u(x,\tau) + 2k_m^2 \operatorname{sech}^2 k_m(x-p_m)\right| = O(\varepsilon^{1-\frac{1}{2}p}) \;.$$

(Notice, that for $m \geq 2$: $D^+ \cap \{E_m^o \times I\} = E_m^o \times I$.)

Moreover:

$$(6.24)\qquad \sup_{\tau\in[0,A]} |k_n(\tau) - k_n^o(\tau)| = \beta \;,$$

with

$$\beta = \begin{cases} O(\varepsilon^{2-\frac{3}{2}p}) & , \quad \frac{2}{3} < p \leq 1 \;, \\ O(\varepsilon) & , \quad 0 \leq p \leq \frac{2}{3} \;, \quad b(k,0) \equiv 0 \;, \\ O(\varepsilon \log \frac{1}{\varepsilon}) & , \quad 0 \leq p \leq \frac{2}{3} \;, \quad b(k,0) \not\equiv 0 \;. \end{cases}$$

So, each soliton can be approximated in the following way:

$$(6.25)\qquad 2k_m^2 \operatorname{sech}^2 k_m \tilde{z}_m = 2\left((k_m^o)^2 + \beta\right) \cdot$$

$$\cdot \operatorname{sech}^2\left[(k_m^o + \beta)\left(x - p_n^o(\tau) + O(\beta(1+\tau\varepsilon^{-p}))\right)\right] \;.$$

Situation $b(k,0) \equiv 0$:

The only difference between this situation and the situation $b(k,0) \not\equiv 0$ is, that in (6.22a) we can take $\mathbb{R} \times [0,A]$ instead of D^+, whilst in (6.23a), we can take D instead of D^+.

For a physical interpretation of the results on the $1/\varepsilon$-timescale, two aspects are of interest. First, the difference in shape and position between a soliton of the pKdV with initial function $U(x)$ and the corresponding soliton of the KdV with the same initial function. Second, the difference in shape and position between a soliton of the pKdV and its approximation.
As to the first aspect, we conclude that, generally, the effect of the perturbation will be considerable. Namely, a soliton of the KdV is given by

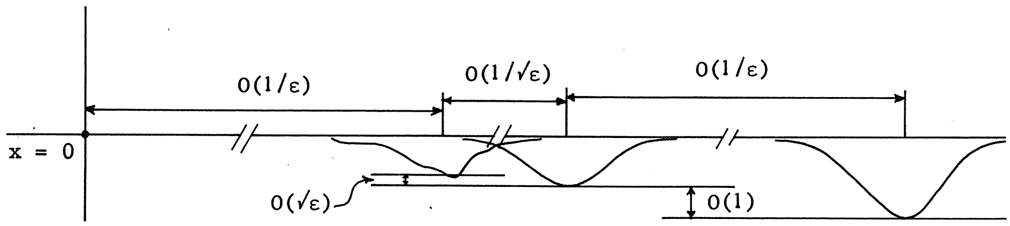
$2\kappa_n^2 \operatorname{sech}^2 \kappa_n(x - 4\kappa_n^2 t)$, while the corresponding soliton of the pKdV is given by $2k_n^2(\tau) \operatorname{sech}^2 k_n(\tau)(x - p_n(\tau))$. From the evolution equation of $k_n(\tau)$ it can be seen that the difference between $k_n(\tau)$ and κ_n will generally become of $O(1)$ on the $1/\varepsilon$-timescale. Accordingly, the difference in shape and the relative difference in position will become of $O(1)$ too. The absolute difference in position will become of $O(1/\varepsilon)$.

Regarding the second aspect, we have: The difference in shape is of $O(\varepsilon^{\frac{1}{2}})$. The difference in position seems, at first sight, to be unsatisfactory. Namely, $p_n(\tau) - p_n^o(\tau) = O(\varepsilon^{\frac{1}{2}} + \varepsilon^{-\frac{1}{2}}\tau)$. However, it should be realized, that it is not the absolute fault but the relative fault in the position of the soliton that determines whether the approximation is satisfactory or not. This relative fault is of $O(\varepsilon^{3/2} + \varepsilon^{\frac{1}{2}}\tau)$.

Summarizing, we have that on the $1/\varepsilon$-timescale, the maximal fault in shape as well as in position is of $O(\varepsilon^{\frac{1}{2}})$.

Illustrating both aspects in a picture, we get:

Generic situation for $t = \frac{1}{\varepsilon}$.



In this picture the left-hand curve represents a soliton of the pKdV. The curve in the middle represents its approximation. The right-hand curve represents the corresponding KdV-soliton. (Of course, the choice of this order is arbitrary.)

CHAPTER VII
EXAMPLES, APPLICATIONS AND EXTENSIONS

VII.1. A trivial but illustrative example, $f(u) = u_{xxx}$

Consider:

(7.1.1) $$\begin{cases} u_t - 6uu_x + u_{xxx} = \varepsilon u_{xxx} , \quad \varepsilon > 0 , \\ u(x,0) = -2 \operatorname{sech}^2 x . \end{cases}$$

By the change of variables:

(7.1.2) $$\hat{x} = x(1-\varepsilon)^{-\frac{1}{2}} \; ; \quad \hat{t} = t(1-\varepsilon)^{-\frac{1}{2}} \; ; \quad \hat{u}(\hat{x},\hat{t}) = u(x,t) ,$$

this equation changes into an initial value problem for the KdV:

(7.1.3) $$\begin{cases} \hat{u}_{\hat{t}} - 6\hat{u}\hat{u}_{\hat{x}} + \hat{u}_{\hat{x}\hat{x}\hat{x}} = 0 , \\ \hat{u}(\hat{x},0) = -2 \operatorname{sech}^2 (1-\varepsilon)^{\frac{1}{2}} \hat{x} . \end{cases}$$

In, for instance, [F], the spectral data were calculated for potentials of the form:

(7.1.4) $$u(x) = -A \operatorname{sech}^2 \alpha x , \quad A > 0, \quad \alpha > 0 .$$

With $n \in \mathbb{R}$ defined by

(7.1.5) $$A = \alpha^2 n(n-1) , \quad n > 1 , \quad n \in \mathbb{R} ,$$

the eigenvalues $\lambda_m = -k_m^2$ are given by:

(7.1.6) $$k_m = \alpha\{n - ([n] + 1 - m)\} , \quad m = 1,\dots,[n] .$$

To solve (7.1.3) by IST, we must calculate the eigenvalues of:

(7.1.7) $$-2 \operatorname{sech}^2 (1-\varepsilon)^{\frac{1}{2}} \hat{x} = -(1-\varepsilon)n(n-1) \operatorname{sech}^2 (1-\varepsilon)^{\frac{1}{2}} \hat{x} ,$$

$$\text{for } n = \tfrac{1}{2} + \tfrac{1}{2}(1 + \tfrac{8}{1-\varepsilon})^{\frac{1}{2}} .$$

They are given by

(7.1.8) $\hat{k}_1 = \alpha(n-2) = -\frac{3}{2}\sqrt{1-\varepsilon} + \frac{1}{2}\sqrt{9-\varepsilon} = \frac{2}{3}\varepsilon + O(\varepsilon^2)$,

$\hat{k}_2 = \alpha(n-1) = -\frac{1}{2}\sqrt{1-\varepsilon} + \frac{1}{2}\sqrt{9-\varepsilon} = 1 + \frac{1}{6}\varepsilon + O(\varepsilon^2)$.

Since $\hat{u}(\hat{x},0)$ decays exponentially for $|\hat{x}| \to \infty$, we can use (4.1.53). Combining this with (3.2.49) leads to

(7.1.9)
$$\lim_{\substack{|\bar{\hat{x}}|=|\hat{x}-4c^2\hat{t}|\leq M \\ \hat{t}\to\infty}} \bar{\hat{u}}(\bar{\hat{x}},\hat{t}) = \begin{cases} 0 & , \text{ if } c \neq \hat{k}_1, \hat{k}_2 , \\ -2\hat{k}_1^2 \operatorname{sech}^2 \hat{k}_1 \bar{\hat{x}} , & \text{if } c = \hat{k}_1 , \\ -2\hat{k}_2^2 \operatorname{sech}^2 \hat{k}_2 \bar{\hat{x}} , & \text{if } c = \hat{k}_2 . \end{cases}$$

Or, in (x,t) variables:

(7.1.10)
$$\lim_{\substack{|\bar{x}|=|x-4c^2 t|\leq M \\ t\to\infty}} \bar{u}(\bar{x},t) =$$

$$= \begin{cases} 0 & , \quad c \neq \hat{k}_1, \hat{k}_2 , \\ -2\hat{k}_1^2 \operatorname{sech}^2 \hat{k}_1 \dfrac{\bar{x}}{\sqrt{1-\varepsilon}} = O(\varepsilon^2) & , \quad c = \hat{k}_1 , \\ -2\hat{k}_2^2 \operatorname{sech}^2 \hat{k}_2 \dfrac{\bar{x}}{\sqrt{1-\varepsilon}} = -2 \operatorname{sech}^2 \bar{x} + O(\varepsilon) & , \quad c = \hat{k}_2 . \end{cases}$$

Now, we calculate the soliton approximation on the $1/\varepsilon$-timescale for the solution of (7.1.1) with the perturbation scheme:
$u(x,0) = -2 \operatorname{sech}^2 x$ has only one eigenvalue $\lambda_1 = -1$. So, $k_1^o(\tau)$ is the solution of:

(7.1.11)
$$\begin{cases} \dfrac{d}{d\tau} k_1^o(\tau) = -\frac{1}{4} \displaystyle\int_{-\infty}^{\infty} \dfrac{d^3}{dx^3} \left(-2(k_1^o)^2 \operatorname{sech}^2 k_1^o x\right) \operatorname{sech}^2 k_1^o x \, dx , \\ k_1^o(0) = 1 . \end{cases}$$

Since $\operatorname{sech}^2 k_1^o x$ is an even function, it is obvious that:

(7.1.12) $k_1^o(\tau) = 1$.

Using (6.23,24) we get:

(7.1.13) $$\sup_{\mathbb{R}\times[m\varepsilon \log\frac{1}{\varepsilon}, A]} |u(x,\tau) - 2\,\mathrm{sech}^2\,[x - \tfrac{1}{\varepsilon}(4\tau + O(\sqrt{\varepsilon}))]| = O(\sqrt{\varepsilon}) .$$

Indeed, this result agrees with (7.1.10).

We make the following observations:

Remarks (7.1.1):

1°. Apparently, the solution $\hat{u}(\hat{x},\hat{t}) = u(x,t)$ contains two solitons. This does not contradict the assumption that no new eigenvalues will be created, since the eigenvalue problems that have been used: $\psi_{xx} + (\lambda - u)\psi = 0$ and $\hat{\psi}_{\hat{x}\hat{x}} + (\lambda - \hat{u})\hat{\psi} = 0$, are not identical.
Moreover, the small soliton is $O(\varepsilon^2)$ for all t and, therefore, is covered by the $O(\sqrt{\varepsilon})$-term in (6.23).

2°. Better perturbation results can be expected if we do not approximate the soliton position by:

$$\frac{4}{\varepsilon}\int_0^{\tau} (k_1^o(\tau'))^2\, d\,' + \frac{1}{2k_1^o(\tau)} \log \frac{c_1^2(0)}{2k_1^o(\tau)} ,$$

but by:

(7.1.14) $$\frac{1}{\varepsilon}\left\{4\int_0^{\tau} (k_1^o(\tau'))^2 d\tau' + \frac{\varepsilon}{k_1^o(\tau)}\int_0^{\tau} \frac{\bar{H}_{1s}^o(\tau')}{2k_1^o(\tau')}\, d\tau'\right\} + \frac{1}{2k_1^o(\tau)} \log \frac{c_1^2(0)}{2k_1^o(\tau)} .$$

Here $\bar{H}_{1s}^o(\tau)$ is defined by:

(7.1.15) i) Replace x, u, ϕ_1, ψ_1 in the definition for $H_1(\tau)$ by $\tilde{z}_1 = x - \varphi_1 - \delta_1$, $\bar{u}(\tilde{z}_1,t)$ $(= u(x,t))$, $\bar{\phi}_1$, $\bar{\psi}_1$.
The expression so formed is defined as $\bar{H}_1(\tau)$.

ii) Replace $\bar{u}$, $\bar{\phi}_1$, $\bar{\psi}_1$ in $\bar{H}_1(\tau)$ by $\bar{u}_s$, $\bar{\phi}_{1s}$, $\bar{\psi}_{1s}$.
The expression so formed is defined as $\bar{H}_{1s}(\tau)$.

iii) Replace $\bar{u}_s$, $\bar{\phi}_{1s}$, $\bar{\psi}_{1s}$ in $\bar{H}_{1s}(\tau)$ by their soliton approximations:

$$\bar{u}_s^o := -2(k_1^o)^2 \operatorname{sech}^2 k_1^o \hat{z}_1 \ ; \quad \bar{\psi}_{1s}^o := \sqrt{\tfrac{1}{2}k_1^o} \operatorname{sech} k_1^o \tilde{z}_1 \ ;$$

$$\bar{\phi}_{1s}^o := -\sqrt{2k_1^o}\left(\tilde{z}_1 + \frac{1}{2k_1^o}\sinh 2k_1^o \tilde{z}_1\right)\operatorname{sech} k_1^o \tilde{z}_1 \ .$$

The expression so formed is defined as $\bar{H}_{1s}^o(\tau)$.

(The eigenfunction ψ_{1s} and generalized eigenfunction ϕ_{1s} of the one-soliton potential

$$u_s = -2\frac{d^2}{dx^2}\log\left(1 + \frac{c_1^2}{2k_1} e^{-2k_1 x}\right),$$

are easily calculated by using the variable

$$\tilde{z}_1 = x - \varphi_1 - \delta_1 = x - \frac{1}{2k_1}\log\frac{c_1^2}{2k_1} \ .$$

We then find

$$u_s(x,t) = \bar{u}_s(\tilde{z}_1,t) = -2k_1^2 \operatorname{sech}^2 k_1 \tilde{z}_1 \Rightarrow \bar{\psi}_{1s}(\tilde{z}_1,t) = \sqrt{\frac{k_1}{2}} \operatorname{sech} k_1 \tilde{z}_1$$

(since $u_s = -4k_1 \psi_{1s}^2$) $\Rightarrow$

$$\bar{\phi}_{1s}(\tilde{z}_1,t) = -2k_1 \bar{\psi}_{1s} \int_0^{\tilde{z}_1} \bar{\psi}_{1s}^{-2}(y,t)\,dy =$$

$$= -\sqrt{2k_1}\left(\tilde{z}_1 + \frac{1}{2k_1}\sinh 2k_1 \tilde{z}_1\right)\operatorname{sech} k_1 \tilde{z}_1 \ .)$$

When calculating $\bar{H}_{1s}^o(\tau)$ we find that:

$$\bar{H}_{1s}^o(\tau) = \int_{-\infty}^{\infty}\left(\bar{\phi}_{1s}^o \bar{\psi}_{1s}^o \int_{-\infty}^{\tilde{z}_1} f(\bar{u}_s^o)\,\bar{\psi}_{1s}^{o\,2}\, d\tilde{z}_1'\right) d\tilde{z}_1 +$$

$$- \int_{-\infty}^{\infty} \bar{\psi}_{1s}^{o\,2}\left(\int_{-\infty}^{\tilde{z}_1} f(\bar{u}_s^o)\bar{\phi}_{1s}^o \bar{\psi}_{1s}^o\, d\tilde{z}_1'\right) d\tilde{z}_1 +$$

$$+ \int_{-\infty}^{\infty} f(\bar{u}_s^o)\bar{\phi}_{1s}^o \bar{\psi}_{1s}^o \, d\tilde{z}_1 = 0 - \frac{8}{3} + \frac{16}{3} = \frac{8}{3} .$$

Since $c_1(0) = \sqrt{2}$, substitution of $k_1^o = 1$, $\bar{H}_{1s}^o = \frac{8}{3}$ and $c_1(0)$ in (7.1.14) leads to:

The position of the soliton is approximated by:

$$\frac{4}{\varepsilon} (\tau + \frac{1}{3} \varepsilon\tau) .$$

This indeed agrees with the real position of the soliton, given by:

$$4\hat{k}_2^2 t = 4(1 + \frac{1}{3}\varepsilon + O(\varepsilon^2))t .$$

This example illustrates the idea that, especially in the 1-soliton case, better approximation of the soliton positions can be given by:

$$(7.1.16) \quad \frac{4}{\varepsilon} \int_0^{\tau} k_n^o(\tau')^2 d\tau' + \frac{1}{k_n^o(\tau)} \int_0^{\tau} \frac{\bar{H}_{ns}^o(\tau')}{2k_n^o(\tau')} d\tau' +$$

$$+ \frac{1}{2k_n^o(\tau)} \log \left\{ \frac{c_n^2(0)}{2k_n^o(\tau)} \prod_{i=n+1}^{N} \left(\frac{k_n^o(\tau) - k_i^o(\tau)}{k_n^o(\tau) + k_i^o(\tau)} \right)^2 \right\} .$$

Proving this concept, comes down on showing that:

$$(7.1.17) \quad \text{a)} \quad \int_0^{\tau} \bar{H}_n(\tau') - \bar{H}_{ns}(\tau')d\tau' = o(1) \quad , \quad \tau \in [0,A] ,$$

$$\text{b)} \quad \int_0^{\tau} \bar{H}_{ns}(\tau') - \bar{H}_{ns}^o(\tau')d\tau' = o(1) \quad , \quad \tau \in [0,A] .$$

Because of the expressions with $\phi_n\psi_n$ in $H_n(\tau)$, we cannot prove (7.1.17) in a way analogously to the proofs of Lemmae (6.1,2,3).

In the case of one-soliton and $_{-\infty}\int^{\infty} f(u_s)\psi_{ns}^2 dx = 0$, the problem is considerably simplified. However, proving (7.1.17a) is still far from trivial.

VII.2. Pure polynomial perturbations

In this section, we will consider the case that f(u) is a polynomial.

(7.2.1) $$f(u) = \sum_{\ell=0}^{q} a_\ell \prod_{s=0}^{j_\ell} \left(\frac{\partial^s u}{\partial x^s}\right)^{p_{s\ell}} , \quad a_\ell \neq 0 , \quad p_{s\ell} \in \mathbb{N} \cup \{0\} , \quad \prod_{s=0}^{j_\ell} p_{s\ell} \neq 0 .$$

For the eigenvalue approximation $k_n^o(\tau)$, we then have:

(7.2.2) $$\frac{d}{d\tau} k_n^o(\tau) = - \frac{\varepsilon}{4\delta(\varepsilon)} \cdot$$

$$\cdot \sum_{\ell=0}^{q} a_\ell \int_{-\infty}^{\infty} \left\{ \prod_{s=0}^{j_\ell} \left[\frac{\partial^s}{\partial x^s} \left(-2(k_n^o)^2 \operatorname{sech}^2 k_n^o x\right)\right]^{p_{s\ell}} \right\} \operatorname{sech}^2 k_n^o x \, dx =$$

$$= - \frac{\varepsilon}{4\delta(\varepsilon)} \sum_{\ell=0}^{q} \alpha_\ell \left(k_n^o(\tau)\right)^{\beta_\ell} ,$$

with

(7.2.3) a) $$\alpha_\ell = a_\ell (-2)^{\sum_{s=0}^{j_\ell} p_{s\ell}} \int_{-\infty}^{\infty} \left[\prod_{s=0}^{j_\ell} \left(\frac{\partial^s}{\partial x^s} \operatorname{sech}^2 x\right)^{p_{s\ell}} \right] \operatorname{sech}^2 x \, dx ;$$

b) $$\beta_\ell = -1 + \sum_{s=0}^{j_\ell} (2+s) p_{s\ell} .$$

Since $\operatorname{sech}^2 x$ is an even function, it immediately follows that:

(7.2.4) $$\sum_{s=0}^{j_\ell} s p_{s\ell} \text{ is odd} , \quad \ell = 0,\ldots,q \Rightarrow \frac{d}{d\tau} k_n^o(\tau) = 0$$

$$\Rightarrow k_n^o(\tau) = \kappa_n \Rightarrow p_n^o(\tau) = \frac{4}{\delta(\varepsilon)} \kappa_n^2 \tau + \frac{1}{2k_n} \log \left\{ \frac{c_n^o(0)}{2k_n} \prod_{i=n+1}^{N} \left(\frac{k_n - k_i}{k_n + k_i}\right)^2 \right\} .$$

We will now investigate the situation that f(u) consists of only one 'term':

(7.2.5) $$f(u) = a \prod_{s=0}^{j} \left(\frac{\partial^s u}{\partial x^s}\right)^{p_s} .$$

We have:

(7.2.6) a) $\frac{d}{d\tau} k_n^o(\tau) = 0$ iff $\sum_{s=0}^{j} sp_s$ is odd ;

b) $\frac{d}{d\tau} k_n^o(\tau) = - \frac{\varepsilon}{4\delta(\varepsilon)} \alpha (k_n^o)^\beta$,

with

(7.2.7) a) $\alpha = a(-2)^{\sum_{s=0}^{j} sp_s} \int_{-\infty}^{\infty} \left[\prod_{s=0}^{j} \left(\frac{\partial^s}{\partial x^s} \operatorname{sech}^2 x \right)^{p_{s\ell}} \right] \operatorname{sech}^2 x \, dx$;

b) $\beta = -1 + \sum_{s=0}^{j} (2+s)p_s$ is odd (i.e. $\sum_{s=0}^{j} sp_s$ is even) .

We note that:

(7.2.8) $\beta = 1 \Leftrightarrow p_0 = 1 \; ; \; p_s = 0 \, , \; s = 1,\ldots,j \Leftrightarrow f(u) = au \quad (\alpha = \frac{4}{3} a)$.

Integration of (7.2.6b) with $k_n^o(\tau) = \kappa_n$ gives:

(7.2.9) a) $k_n^o(\tau) = \kappa_n \, e^{-\frac{a\varepsilon}{3\delta(\varepsilon)}\tau}$ iff $f(u) = au$;

b) $k_n^o(\tau) = \left[\frac{\alpha\varepsilon\tau}{4\delta(\varepsilon)} (\beta - 1) + \kappa_n^{1-\beta} \right]^{\frac{1}{1-\beta}}$, $\beta = 3,5,7,\ldots$.

We can apply these results to, for example, the 'Korteweg-de Vries-εBurgers equation':

(7.2.10) $u_t - 6uu_x + u_{xxx} = \pm \varepsilon u_{xx}$.

With (7.2.7,9) we find

(7.2.11) a) $\beta = 3 \, , \; \alpha = \pm 4 \int_{-\infty}^{\infty} \left(\frac{\partial^2}{\partial x^2} \operatorname{sech}^2 x \right) \operatorname{sech}^2 x \, dx = \mp 4 \cdot \frac{32}{25}$;

b) $k_n^o(\tau) = \left[\mp \frac{64}{15} \frac{\varepsilon}{\delta(\varepsilon)} \tau + \kappa_n^{-2} \right]^{-\frac{1}{2}}$.

For the approximation of the soliton positions this gives:

(7.2.12) $p_n^o(\tau) = \frac{4}{\delta(\varepsilon)} \int_0^\tau \left(k_n^o(\tau') \right)^2 d\tau' = \left| \frac{1}{\varepsilon} \frac{15}{16} \log \left(1 \mp \kappa_n^2 \frac{64}{15} \frac{\varepsilon}{\delta(\varepsilon)} \tau \right) \right|$.

If (7.2.10) has a − sign, then we have consistency with respect to (3.2.3b) for arbitrary compacta on the $1/\varepsilon$-timescale. If, however, (7.2.10) has a + sign, then we only have consistency with respect to (3.2.3b) on the $1/\varepsilon$-timescale for $\tau \in [0,A]$ with $A > \frac{15}{64}\kappa_N^2$.

VII.3. The shallow water wave perturbation, $f(u) = \frac{3}{2}u^2u_x + \frac{5}{2}uu_{xxx} + \frac{23}{4}u_xu_{xx} - \frac{19}{40}u_{xxxxx}$

In this section we consider:

$$(7.3.1) \qquad u_t - 6uu_x + u_{xxx} = \varepsilon\{\frac{3}{2}u^2u_x + \frac{5}{2}uu_{xxx} + \frac{23}{4}u_xu_{xx} - \frac{19}{40}u_{xxxxx}\} = \\ =: \varepsilon f(u) \; .$$

The motivation for looking at (7.3.1) is the following: In modelling so called 'shallow-waterwaves', two small parameters play a role, namely

$$\alpha = \frac{a}{h_0} \; ; \quad \beta = \frac{h_0^2}{\ell^2} \; ;$$

a : typical wave amplitude
ℓ : typical wave length
h_0: depth of water in rest

Taking α and β to have the same order of magnitude, the number of significant small parameters is reduced to one, called ε. When carrying out a formal expansion in ε, the KdV-equation will be found as the lowest order term (see [KdV], [W]). Also taking into account first order contributions, leads to (7.3.1). This is shown in Appendix E1.

Since (7.3.1) is obtained from a formal expansion in ε, it is natural to try to find solutions of (7.3.1), in the form of a power series in ε. Inspired by the solitary-wave solutions of the KdV, we substitute:

$$(7.3.2) \qquad u(x,t) = \sum_0^\infty \varepsilon^n u_n(\bar{x}) \; ,$$

where, for $\bar{x}$ and $u_0(\bar{x})$ we take:

$$(7.3.3) \qquad \text{a)} \quad \bar{x} = x - (4\kappa^2 - \varepsilon a\kappa^4)t - x_0 \; ,$$

$$\text{b)} \quad u_0(\bar{x}) = -2\kappa^2 \operatorname{sech}^2 \kappa\bar{x}$$

We will now determine $u_1(\bar{x})$, in such a way that $v(x,t) = u_0(\bar{x}) + \varepsilon u_1(\bar{x})$ satisfies:

$$(7.3.4) \qquad v_t - 6vv_x + v_{xxx} = \varepsilon f(v) + O(\varepsilon^2) \ .$$

For $u_1(\bar{x})$, this leads to:

$$(7.3.5) \qquad -4\kappa^2 u_{1\bar{x}} - 6(u_0 u_1)_{\bar{x}} + u_{1\bar{x}\bar{x}\bar{x}} = f(u_0) - a\kappa^4 u_{0\bar{x}} \ .$$

Integrating once, using

$$\lim_{|\bar{x}|\to\infty} |u_0(\bar{x})| + \varepsilon|u_1(\bar{x})| = 0 \ ,$$

using

$$u_{0\bar{x}\bar{x}\bar{x}\bar{x}} = \frac{d}{d\bar{x}} (4\kappa^2 u_{0\bar{x}} + 6u_0 u_{0\bar{x}}) \ ,$$

and substituting

$$u_0(\bar{x}) = -2\kappa^2 \operatorname{sech}^2 \kappa\bar{x} \ ,$$

gives us:

$$(7.3.6) \qquad u_{1\bar{x}\bar{x}} + (12\kappa^2 \cosh^{-2}\kappa\bar{x} - 4\kappa^2)u_1 =$$

$$= \kappa^6\left(24 \cosh^{-6}\kappa\bar{x} - 48 \cosh^{-4}\kappa\bar{x} + [\tfrac{76}{5} + 2a] \cosh^{-2}\kappa\bar{x}\right) \ .$$

In Appendix E2 the general solution of (7.3.6), satisfying $\lim_{|\bar{x}|\to\infty} u_1(\bar{x}) = 0$, has been found to be given by:

$$(7.3.7) \qquad u_1(\bar{x}) = \kappa^4\{(\tfrac{9}{5} + \tfrac{1}{2}a) \cosh^{-2}\kappa\bar{x} - 3 \cosh^{-4}\kappa\bar{x} +$$

$$- (\tfrac{19}{5} + \tfrac{1}{2}a)\kappa\bar{x} \cosh^{-3}\kappa x \sinh\kappa\bar{x}\} + A \cosh^{-3}\kappa\bar{x} \sinh\kappa\bar{x} \ .$$

It is evident that this method of finding solutions is not suitable for solving initial value problems. In order to see what kind of soliton solutions emerge from a given initial function, we need the perturbation theory.

For the eigenvalue approximations, we find with (7.2.4) that:

$$(7.3.8) \qquad k_n^o(\tau) = \kappa_n \ , \quad p_n^o(\tau) = \frac{4}{\delta(\varepsilon)} \kappa_n^2 \tau \ .$$

This result indicates that, on the $1/\varepsilon$-timescale, the KdV-equation gives a good description of the physical reality in the shallow-waterwave theory. As in § VII.1, better approximations of the soliton positions are given by (7.1.16). We will now calculate $\bar{H}^o_{1s}$ for the one-soliton case.

$$(7.3.9)\quad \bar{H}^o_{1s}(\tau) = \int_{-\infty}^{\infty} \bar{\phi}^o_{1s}\bar{\psi}^o_{1s}\left(\int_{-\infty}^{\tilde{z}_1} f(\bar{u}^o_s)(\bar{\psi}^o_{1s})^2\, d\tilde{z}'_1\right) d\tilde{z}_1 +$$

$$- \int_{-\infty}^{\infty} (\bar{\psi}^o_{1s})^2\left(\int_{-\infty}^{\tilde{z}_1} f(\bar{u}^o_s)\bar{\phi}^o_{1s}\bar{\psi}^o_{1s}\, d\tilde{z}'_1\right) dz_1 + \int_{-\infty}^{\infty} f(\bar{u}^o_s)\bar{\psi}^o_{1s}\bar{\phi}^o_{1s}\, d\tilde{z}_1 \ .$$

Determining $\bar{H}^o_{1s}(\tau)$ is simplified by using:

$$(7.3.10)\quad \text{a)}\quad f \text{ integrable and odd} \Rightarrow \int_{-\infty}^{x} f(t)dt \quad \text{is even},$$

$$\text{b)}\quad f \text{ integrable and even} \Rightarrow \int_{-\infty}^{x} f(t)dt = \tfrac{1}{2}\int_{-\infty}^{\infty} f(t)dt + g(x),$$

$$\text{with}\quad g(x) = \tfrac{1}{2}\int_{-x}^{x} f(t)dt \quad \text{is odd.}$$

Moreover, we integrate by parts to find

$$(7.3.11)\quad \int_{-\infty}^{\infty} (\bar{\psi}^o_{1s})^2\left(\int_{-\infty}^{\tilde{z}_1} f(\bar{u}^o_s)\bar{\phi}^o_{1s}\bar{\psi}^o_{1s}\, d\tilde{z}'_1\right) d\tilde{z}_1 =$$

$$= \int_{-\infty}^{\infty} f(\bar{u}^o_s)\bar{\phi}^o_{1s}\bar{\psi}^o_{1s}\, d\tilde{z}_1 - \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\tilde{z}_1} (\bar{\psi}^o_{1s})^2\, dz'_1\right) f(\bar{u}^o_s)\bar{\phi}^o_{1s}\bar{\psi}^o_{1s}\, d\tilde{z}_1 \ .$$

Using (7.3.9,10,11) and some calculations gives us:

$$(7.3.12)\quad \bar{H}^o_{1s}(\tau) = \tfrac{1}{2}\int_{-\infty}^{\infty} f(\bar{u}^o_s)\bar{\phi}^o_{1s}\bar{\psi}^o_{1s}\, d\tilde{z}_1 = -\kappa_1^6\,\frac{124}{15} \ .$$

So, for the soliton position $p_1(\tau)$ we find

(7.3.13) $p_1(\tau) = \frac{1}{\varepsilon}\,(4\kappa_1^2\tau - \varepsilon\kappa_1^4\frac{62}{15}\tau) + \frac{1}{2\kappa_1}\log\frac{c_1^2(0)}{2\kappa_1} + o(1)$.

We see that, when, in (7.3.3) we take $a = 62/15$, then the soliton approximation $-2\kappa_1^2\operatorname{sech}^2\kappa_1(x - p_1(\tau))$ coincides with $u_0(\bar{x})$ up to first order.
This shows that a combination of the perturbation scheme with the method of finding solutions by inserting a power series in ε can be useful. First, use the perturbation scheme to determine $u_0(\bar{x})$ and, then, use (7.3.2) to determine next order terms.

VII.4. The inadmissible perturbation, $f(u) = u + \frac{1}{2}xu_x$

In this section, we apply the perturbation scheme to the pKdV with perturbation: $\varepsilon(u + \frac{1}{2}xu_x)$. Since this perturbation depends explicitly on x, we must adapt the perturbation scheme in order to obtain useful results.
Consider:

(7.4.1) $$\begin{cases} u_t - 6uu_x + u_{xxx} = \varepsilon(u + \frac{1}{2}xu_x) , \\ u(x,0) = U(x) . \end{cases}$$

As can be seen in Appendix A.1 and (2.1.17a), this problem can be integrated with the inverse scattering method. We will calculate the pure 2-soliton solution explicitly, as well as use a modified formal version of the perturbation scheme to derive an approximation of the 2-soliton solution. Of course, our goal is to show that the approximation obtained from the modified version of the perturbation scheme matches the real solution.

We take:

(7.4.2) $U(x) = -6\operatorname{sech}^2 x$.

For this potential we have:

(7.4.3) $b(k) \equiv 0$; $k_1 = 1$; $c_1 = \sqrt{6}$; $k_2 = 2$; $c_2 = \sqrt{12}$.

With the evolution equations (A.1.20,26,29) for the spectral data, we find:

$$(7.4.4)\qquad b(k,t) \equiv 0\ ;\quad k_1(t) = e^{\frac{1}{2}\varepsilon t}\ ;\quad c_1(t) = \sqrt{6}\,\exp\left\{\tfrac{1}{4}\varepsilon t + \frac{8}{3\varepsilon}\,(e^{\frac{3}{2}\varepsilon t} - 1)\right\}\ ;$$

$$k_2(t) = 2e^{\frac{1}{2}\varepsilon t}\ ;\quad c_2(t) = \sqrt{12}\,\exp\left\{\tfrac{1}{4}\varepsilon t + \frac{64}{3\varepsilon}\,(e^{\frac{3}{2}\varepsilon t} - 1)\right\}\ .$$

The explicit solution of (7.4.1,2) is given by:

$$(7.4.5)\qquad u(x,t) = -2\frac{d^2}{dx^2}\log\det\begin{pmatrix} 1+\dfrac{c_1^2}{2k_1}\,e^{-2k_1x} & \dfrac{c_1c_2}{k_1+k_2}\,e^{-(k_1+k_2)x} \\ \dfrac{c_1c_2}{k_1+k_2}\,e^{-(k_1+k_2)x} & 1+\dfrac{c_2^2}{2k_2}\,e^{-2k_2x} \end{pmatrix} =$$

$$= -12e^{\varepsilon t}\left(\frac{3+\cosh 2\eta + 4\cosh 2\xi}{[\cosh(\eta+\xi) + 3\cosh(\eta-\xi)]^2}\right),$$

with

$$(7.4.6)\qquad \text{a)}\quad \xi = e^{\frac{1}{2}\varepsilon t}x - \frac{8}{3\varepsilon}\,(e^{\frac{3}{2}\varepsilon t} - 1)\ .$$

$$\text{b)}\quad \eta = 2e^{\frac{1}{2}\varepsilon t}x - \frac{64}{3\varepsilon}\,(e^{\frac{3}{2}\ t} - 1)\ .$$

In particular we are interested in the asymptotic behaviour of this solution. In order to be able to compare the results obtained directly from (7.4.5), with those obtained from the perturbation analysis on the $1/\varepsilon$-timescale, we perpetrate $\varepsilon \downarrow 0$ asymptotics on compacta in $\tau = \varepsilon t$.

To obtain asymptotic results from (7.4.5) we use Theorem (3.2.1). We therefore have to show that conditions (3.2.3,13) are satisfied on the $1/\varepsilon$-timescale.

That (3.2.3) is satisfied is easily seen from (7.4.4). As usual, we define $\varphi_n(\tau)$ by: $c_n(\tau) = c_n(0)\,\exp(k_n(\tau)\,\varphi_n(\tau))$. With (7.4.4) this gives:

$$(7.4.7)\qquad \text{a)}\quad \varphi_1(\tau) = \frac{1}{4}\tau e^{-\frac{1}{2}\tau} + \frac{8}{3\varepsilon}\,(e^{\tau} - e^{-\frac{1}{2}\tau})\ ;$$

$$\text{b)}\quad \varphi_2(\tau) = \frac{1}{8}\tau e^{-\frac{1}{2}\tau} + \frac{32}{3\varepsilon}\,(e^{\tau} - e^{-\frac{1}{2}\tau})\ .$$

So, condition (3.2.13) is satisfied too.

For the quantities δ_n^+, defined by (3.2.14a), we find:

(7.4.8) a) $\delta_1^+(\tau) = -\frac{1}{4}\,\tau e^{-\frac{1}{2}\tau} - \frac{1}{2}\,e^{-\frac{1}{2}\tau}\log 3$;

b) $\delta_2^+(\tau) = \frac{1}{4}\,e^{-\frac{1}{2}\tau}\log 3 - \frac{1}{8}\,\tau e^{-\frac{1}{2}\tau}$.

From (7.4.7,8) we obtain:

(7.4.9) a) $k_1\tilde{z}_1 = \xi + \frac{1}{2}\log 3$;

b) $k_2\tilde{z}_2 = \eta - \frac{1}{2}\log 3$.

With Corollary (3.2.49) we now get:

(7.4.10) a) $u(x,\tau) + 2e^{\tau}\,\mathrm{sech}^2(\xi + \frac{1}{2}\log 3) + 8e^{\tau}\,\mathrm{sech}^2(\eta - \frac{1}{2}\log 3) =$

$= O\left(e^{-\alpha\frac{\tau}{\varepsilon}}\right)$, uniformly in x on $\mathbb{R}$, $\tau \in [0,A]$.

b) $u(x,\tau) + 2e^{\tau}\,\mathrm{sech}^2(\xi + \frac{1}{2}\log 3) = O\left(e^{-\alpha\frac{\tau}{\varepsilon}}\right)$,

uniformly in x on $(-\infty, \frac{20}{3\varepsilon}(e^{\tau} - e^{-\frac{1}{2}\tau})]$, $\tau \in [0,A]$.

c) $u(x,\tau) + 8e^{\tau}\,\mathrm{sech}^2(\eta - \frac{1}{2}\log 3) = O\left(e^{-\alpha\frac{\tau}{\varepsilon}}\right)$,

uniformly in x on $[\frac{20}{3\varepsilon}(e^{\tau} - e^{-\frac{1}{2}\tau}), \infty)$, $\tau \in [0,A]$.

Note that:

$$e^{-\alpha\frac{\tau}{\varepsilon}} = O(\varepsilon^{\alpha m}) \ , \quad \text{for } \tau = m\varepsilon \log\frac{1}{\varepsilon} \ .$$

We will now derive an approximation of the solution of (7.4.1,2) by means of a formal perturbation procedure. On the $1/\varepsilon$-timescale, $\tau \in [0,A]$, we expect the solitons to be approximated by:

(7.4.11) $2\left(k_n^o(\tau)\right)^2\,\mathrm{sech}^2 k_n^o(\tau)\,(x - p_n^o(\tau))$, $n = 1,2$,

where k_n^o is the solution of:

$$(7.4.12)\quad \begin{cases} \dfrac{d}{d\tau} k_n(\tau) = -\dfrac{1}{4} \displaystyle\int_{-\infty}^{\infty} f(\bar{u}_{ns}(\tilde{z}_n))\, \mathrm{sech}^2 k_n^0 \tilde{z}_n \, d\tilde{z}_n \; + \\[2ex] \qquad\qquad - \dfrac{1}{8} (\varphi_n + \delta_n^+) \displaystyle\int_{-\infty}^{\infty} \dfrac{d}{d\tilde{z}_n} (\bar{u}_{ns}(\tilde{z}_n))\, \mathrm{sech}^2 k_n^o \tilde{z}_n \, d\tilde{z}_n \, , \\[2ex] k_n^o(0) = k_n(0) \, , \end{cases}$$

and $\bar{u}_{ns}(\tilde{z}_n)$ is defined by:

$$(7.4.13)\quad \bar{u}_{ns}(\tilde{z}_n) = -\, 2\left(k_n^o\right)^2 \mathrm{sech}^2 k_n^o \tilde{z}_n \; .$$

We have:

$$(7.4.14)\quad \text{a)} \quad \int_{-\infty}^{\infty} f(\bar{u}_{ns})\, \mathrm{sech}^2 k_n^o \tilde{z}_n \, d\tilde{z}_n = -\, 2k_n^o(\tau) \; ;$$

$$\text{b)} \quad \int_{-\infty}^{\infty} \frac{d}{d\tilde{z}_n} (\bar{u}_{ns})\, \mathrm{sech}^2 k_n^o \tilde{z}_n \, d\tilde{z}_n = 0 \quad \text{(integrand is odd)}.$$

So, we find:

$$(7.4.15)\quad k_1^o(\tau) = e^{\frac{1}{2}\tau} \; , \quad k_2^o(\tau) = 2e^{\frac{1}{2}\tau} \; .$$

Finding an approximation of the soliton positions $p_n^o(\tau)$ requires the terms of leading order in $\varphi_n(\tau)$ to be determined. For perturbations that are not depending explicitly on x, the leading order term in $\varphi_n(\tau)$ is given by:

$$\frac{4}{\varepsilon} \int_0^{\tau} k_n^2(\tau')d\tau' \quad \text{(see Lemma (3.2.1))}.$$

We will show that for the perturbation considered here, the leading order term in $\varphi_n(\tau)$ is given by:

$$(7.4.16)\quad \frac{4}{\varepsilon k_n(\tau)} \int_0^{\tau} k_n^3(\tau')d\tau' \; .$$

That is, we have to show that the terms of leading order in $H_n(\tau)$ cancel out.

We can determine the leading order terms in H_n in the same way as was done in § III.2. That is: The leading order terms in H_n are those terms that generate $O(\varphi(\tau))$-contributions, when in the definition of H_n, we replace x by $\bar{x} = x - \varphi$, x' by $\bar{x}' = x' - \varphi$, $\bar{u}(\bar{x},\tau) = u(x,\tau)$, etc.
Carrying out the above substitution we find:

(7.4.17) $$\int_{-\infty}^{\infty} f(u)\psi_n^2\,dx = \int_{-\infty}^{\infty} f(\bar{u})\bar{\psi}_n^2\,d\bar{x} + \tfrac{1}{2}\varphi \int_{-\infty}^{\infty} \bar{u}_{\bar{x}}\bar{\psi}_n^2\,d\bar{x} \;;$$

$$\int_{-\infty}^{x} f(u)\psi_n^2\,dx = \int_{-\infty}^{\bar{x}} f(\bar{u})\bar{\psi}_n^2\,d\bar{x}' + \tfrac{1}{2}\varphi \int_{-\infty}^{\bar{x}} \bar{u}_{\bar{x}'}\bar{\psi}_n^2\,d\bar{x}' \;;$$

$$\int_{-\infty}^{\infty} f(u)\phi_n\psi_n\,dx = \int_{-\infty}^{\infty} f(\bar{u})\bar{\phi}_n\bar{\psi}_n\,d\bar{x} + \tfrac{1}{2}\varphi \int_{-\infty}^{\infty} \bar{u}_{\bar{x}}\,\bar{\phi}_n\bar{\psi}_n\,d\bar{x} \;;$$

$$\int_{-\infty}^{x} f(u)\phi_n\psi_n\,dx = \int_{-\infty}^{\bar{x}} f(\bar{u})\bar{\phi}_n\bar{\psi}_n\,d\bar{x}' + \tfrac{1}{2}\varphi \int_{-\infty}^{\bar{x}} \bar{u}_{\bar{x}'}\,\bar{\phi}_n\bar{\psi}_n\,d\bar{x}' \;;$$

$$\int_{-\infty}^{\infty} x\psi_n^2\,dx = \int_{-\infty}^{\infty} \bar{x}\bar{\psi}_n^2\,d\bar{x} + \varphi \;;$$

$$\theta_n = \lim_{\bar{x}\to\infty}\left\{ \int_{-\infty}^{\bar{x}} (\bar{\phi}_n\bar{\psi}_n - 1)\,d\bar{x}' + 2\bar{x}\right\} + 2\varphi \,.$$

The other integrals in $H_n(\tau)$ do not change.

As before, we expect the functions $\bar{u}$, $\bar{\psi}_n$, $\bar{\phi}_n$ to be approximated by the pure 1-soliton quantities:

(7.4.18) $$\bar{u}_{ns} = -\,2k_n^2\,\mathrm{sech}^2 k_n\bar{x} \;, \quad n = 1, \text{ respectively } n = 2 \;;$$

$$\bar{\psi}_{ns} = \sqrt{\tfrac{1}{2}k_n}\;\mathrm{sech}\,k_n\bar{x} \quad , \quad n = 1,2 \;;$$

$$\bar{\phi}_{ns} = -\sqrt{2k_n}\left(\bar{x} + \frac{1}{2k_n}\sinh 2k_n\bar{x}\right)\mathrm{sech}\,k_n\bar{x} \;, \quad n = 1,2 \,.$$

Calculating the relevant integrals, we find:

(7.4.19) $$\int_{-\infty}^{\infty} f(\bar{u}_{ns})\bar{\psi}_{ns}^2 \, d\bar{x} = -k_n^2 \; ;$$

$$\int_{-\infty}^{\infty} \frac{d}{d\bar{x}}(\bar{u}_{ns})\bar{\psi}_{ns}^2 \, d\bar{x} = 0 \quad \text{(integrand is odd)} \; ;$$

$$\int_{-\infty}^{\infty} \bar{\phi}_{ns}\bar{\psi}_{ns}\left(\int_{-\infty}^{\bar{x}} \frac{d}{d\bar{x}'}(\bar{u}_{ns})\bar{\psi}_{ns}^2 \, d\bar{x}'\right) d\bar{x} = 0 \quad \text{(integrand is odd)} \; ;$$

$$\int_{-\infty}^{\infty} \frac{d}{d\bar{x}}(\bar{u}_{ns})\bar{\phi}_{ns}\bar{\psi}_{ns} = -4k_n^2 \; ;$$

$$\int_{-\infty}^{\infty} \bar{\psi}_{ns}^2\left(\int_{-\infty}^{\bar{x}} \frac{d}{d\bar{x}'}(\bar{u}_{ns})\bar{\phi}_{ns}\bar{\psi}_{ns} \, d\bar{x}'\right) d\bar{x} =$$

$$= \tfrac{1}{2}\left(\int_{-\infty}^{\infty} \bar{\psi}_{ns}^2 \, d\bar{x}\right)\left(\int_{-\infty}^{\infty} \frac{d}{d\bar{x}}(\bar{u}_{ns})\bar{\phi}_{ns}\bar{\psi}_{ns} \, d\bar{x}\right) = -2k_n^2 \; .$$

Now, using (3.1.11,13) and (7.4.17,19), we see that the leading terms in $H_n(\tau)$ add up to:

$$-\tfrac{1}{2}\varphi \cdot -2k_n^2 + \varphi \cdot -k_n^2 + \tfrac{1}{2}\varphi \cdot -4k_n^2 - 2\varphi \cdot -k_n^2 = 0 \; .$$

It follows that the leading part in $\varphi_n(\tau)$, and so in $p_n(\tau)$, is given by:

(7.4.20) $$p_n^o(\tau) = \frac{4}{\varepsilon k_n^o(\tau)} \int_0^{\tau} \left(k_n^o(\tau')\right)^3 d\tau' \; .$$

Using (7.4.15) we find:

(7.4.21) a) $$p_1^o(\tau) = \frac{8}{3\varepsilon}\left(e^{\tau} - e^{-\frac{1}{2}\tau}\right) \; ;$$

b) $$p_2^o(\tau) = \frac{32}{3\varepsilon}\left(e^{\tau} - e^{-\frac{1}{2}\tau}\right) \; .$$

Indeed, we see that the so derived approximations are in complete accordance with the explicitly determined asymptotic behaviour of $u(x,\tau)$.

APPENDIX A

A.1. Derivation of (2.1.14) and (2.1.15); Evolution of the spectral data for solutions of these equations; The solitary wave solutions of (2.1.14) and (2.1.17)

We consider:

(A.1.1) a) $\psi_{xx} + (\lambda + u)\psi = 0$;

b) $\psi_t = A\psi + B\psi_x$.

Putting $\psi_{xxt} = \psi_{txx}$ leads to the following equations for A and B:

(A.1.2) a) $2A_x + B_{xx} = 0$;

b) $A_{xx} - 2B_x(\lambda + u) - Bu_x = - u_t - \lambda_t$

$\Leftrightarrow$

(A.1.3) a) $A = - \frac{1}{2} B_x + \alpha(t)$;

b) $\frac{1}{2} B_{xx} + 2B_x(\lambda + u) + Bu_x = u_t + \lambda_t$.

First we investigate the case that:

(A.1.4) $\lambda_t = 0$.

We try to find a solution of (A.1.3b) by substituting a truncated power series in λ:

(A.1.5) $B(x,t) = \sum_{n=0}^{N} \lambda^n B_n(x,t)$.

Substitution gives:

(A.1.6) $$u_t = \sum_{n=0}^{N} (\tfrac{1}{2}B_n''' + 2B_n' u + u_x B_n)\,\lambda^n + 2B_n'\,\lambda^{n+1} \ , \quad ' = \frac{\partial}{\partial x}$$

$\Leftrightarrow$

(A.1.7) a) $B_N(x,t) = \beta_N(t)$;

b) $$B_n = -\tfrac{1}{4}B''_{n+1} - uB_{n+1} + \tfrac{1}{2}\int_{-\infty}^{x} u_y(y,t)B_{n+1}(y,t)dy + \beta_n(t) \ ,$$

$0 \leqq n \leqq N-1$;

c) $\tfrac{1}{2}B_0''' + 2uB_0' + u_x B_0 = u_t$.

Starting with (A.1.7a), we use the recurrency relation (A.1.7b) to find an expression for $B_0(x,t)$ with the following structure:

(A.1.8) $$B_0(x,t) = \beta_0(t) + \sum_{n=1}^{N} \beta_n(t)(T_n u)(x,t) \ ,$$

where T_n are differential-integral operators.

Substitution of (A.1.8) in (A.1.7c) produces an evolution equation for $u(x,t)$.

Examples:

1°. $N = 1 \Rightarrow u_t + 6uu_x + u_{xxx} = 0$, KdV-equation;

2°. $$N = 2 \Rightarrow \begin{cases} B_2 = \beta_2 \\ B_1 = -\tfrac{1}{2}\beta_2 u + \beta_1 \\ B_0 = \frac{1}{8}\beta_2 u_{xx} + \frac{3}{8}\beta_2 u^2 - \frac{1}{2}\beta_1 u + \beta_0 \ . \end{cases}$$

Substitution in (A.1.7c) and choosing:

$$\beta_0 = 0 \ , \quad \beta_1 = 4 \ , \quad \beta_2 = 16\beta \ ,$$

leads to:

(A.1.9) $$u_t + 6uu_x + u_{xxx} = \beta\{u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x\} \ .$$

The evolution of the 'eigenfunctions' $\psi(x,k,t)$ in time, is given by (A.1.1b), with:

(A.1.10) a) $A = u_x + \alpha + \beta(-u_{xxx} - 6uu_x + 4\lambda u_x)$;

b) $B = -2u + 4\lambda + \beta(2u_{xx} + 6u^2 - 8\lambda u + 16\lambda^2)$.

For determining the time-evolution of the normalization coefficients $c_n(t)$, we take in (A.1.1b): $\psi = \psi_n(x,t)$ and $\lambda = \lambda_n = -k_n^2$. Multiplying (A.1.1b) by ψ_n and integrating over the real axis gives:

$$(A.1.11)\quad 0 = \tfrac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\psi_n^2(x,t)dx = \int_{-\infty}^{\infty}A_n\psi_n^2\,dx + \int_{-\infty}^{\infty}B_n\psi_n\psi_{nx}\,dx =$$

$$= \int_{-\infty}^{\infty}(A_n\psi_n^2 - \tfrac{1}{2}B_n'\psi_n^2)dx = 2\int_{-\infty}^{\infty}A_n\psi_n^2\,dx - \alpha_n =$$

$$= \alpha_n + 2(1+4\lambda_n\beta)\int_{-\infty}^{\infty}u_x\psi_n^2\,dx - 2\beta\int_{-\infty}^{\infty}(u_{xxx} + 6uu_x)\psi_n^2\,dx \ .$$

With the help of the S.E. (A.1.1a), it is simple to deduce that:

(A.1.12) a) $u_x\psi^2 = \frac{d}{dx}(\psi_x^2 - \psi\psi_{xx})$;

b) $(u_{xxx} + 6uu_x)\psi^2 = \frac{d}{dx}(u_{xx}\psi^2 - 2u_x\psi\psi_x + 2u\psi_x^2 + 2u\psi\psi_{xx} + 4\psi_{xx}^2 + 8\lambda\psi_x\psi_{xx} - 4\lambda\psi_x^2)$.

Using these equalities in (A.1.11) leads to:

(A.1.13) $\alpha_n = 0$.

Hence, the evolution of the eigenfunction $\psi_n(x,t)$ is given by:

$$(A.1.14)\quad \frac{\partial}{\partial t}\psi_n = [(-2u - 4k_n^2) + \beta(2uu_x + 6u^2 + 8uk_n^2 + 16k_n^4)]\frac{\partial\psi_n}{\partial x} + [u_x + \beta(-u_{xxx} - 6uu_x - 4k_n^2u_x)]\psi_n \ .$$

For the asymptotic behaviour of ψ_n we have

(A.1.15) $\lim_{x\to\infty} \psi_n e^{k_n x} = c_n(t)$; $\lim_{x\to\infty} \psi_{nx} e^{k_n x} = -k_n c_n(t)$;

$$\lim_{x\to\infty} \psi_{nt} e^{k_n x} = \frac{dc_n}{dt} .$$

Taking limit $x \to \infty$ in (A.1.14), now leads to:

(A.1.16) $$\frac{dc_n}{dt} = (4k_n^3 - 16\beta k_n^5)c_n \Rightarrow c_n(t) = c_n(0)e^{(4k_n^3-16\beta k_n^5)t} .$$

The evolution of the reflection coefficient is determined more simply. For the generalized eigenfunction $\psi(x,k,t)$ we have:

(A.1.17) $\psi(x,k,t) \sim e^{-ikx} + be^{ikx}$, $x \to \infty$;

$\psi_x(x,k,t) \sim -ik e^{-ikx} + bik e^{ikx}$, $x \to \infty$;

$\psi_t(x,k,t) \sim b_t e^{ikx}$, $x \to \infty$.

Substituting $\psi(x,k,t)$ and (A.1.10) into (A.1.1b) and taking limit $x \to \infty$ gives:

(A.1.18) a) $\alpha = 4ik^3 + 16i\beta k^5$;

b) $b_t = (8ik^3 + 32i\beta k^5)b \Rightarrow b(k,t) = b(k,0)e^{(8ik^3+32i\beta k^5)t}$.

With regard to applying inverse scattering, it is important to note that $b(k,0) = 0$ implies $b(k,t) = 0$ for all t.
We now know that (A.1.9) has pure N-soliton solutions. The solitary wave solutions of (A.1.9) are given by:

(A.1.19) $$u(x,t) = 2\kappa^2 \operatorname{sech}^2 \kappa(x - (4\kappa^2 - 16\beta\kappa^4)t) .$$

We return to the equations (A.1.3), but now we take:

(A.1.20) $$\lambda_t = f(\lambda) .$$

First, we derive an evolution equation for $c_n(t)$.
Substituting $\psi_n(x,t)$ into (A.1.1b) and taking limit $x \to \infty$ leads to:

(A.1.21) $$-x\frac{dk_n}{dt}c_n + \frac{dc_n}{dt} = (-\tfrac{1}{2}B'_{n,\infty} + \alpha_n - B_{n,\infty}k_n)c_n ,$$

where $B'_{n,\infty}$, respectively, $B_{n,\infty}$, stand for the asymptotic behaviour of B'_n and B_n, respectively, for $x \to \infty$.

It is obvious that the x-dependent terms in (A.1.21) must cancel out. So:

(A.1.22) $$x\frac{dk_n}{dt} - \tfrac{1}{2}B'_{n,\infty} - B_{n,\infty}k_n \quad \text{is x-independent.}$$

This can easily be established by taking for $B_{n,\infty}$ a linear function in x:

(A.1.23) $$B_{n,\infty}(x,t) = b_0(t) + xb_1(t) ,$$

with

$$b_1(t) = \frac{1}{k_n}\frac{dk_n}{dt} = \frac{1}{2\lambda_n}f(\lambda_n) , \quad b_0(t) \text{ arbitrary.}$$

We will now restrict ourselves to B's of the following form:

(A.1.24) $$B(x,t) = b_0^{(\lambda)}(t) + \frac{x}{2\lambda}f(\lambda) + N^{(\lambda)}u ,$$

where $N^{(\lambda)}u$ satisfies

(A.1.25) $$\lim_{x\to\infty} (N^{(\lambda)}u)(x) = 0 .$$

So, for the evolution equation for $c_n(t)$ we have:

(A.1.26) $$\frac{dc_n}{dt} = \left(\frac{f(\lambda_n)}{4\lambda_n} + \int_{-\infty}^{\infty}\left(N^{(\lambda_n)}u\right)_x \psi_n^2\,dx - b_0^{(\lambda_n)}k_n\right)c_n .$$

Now, we need to derive an evolution equation for b(k,t). Due to the structure of B(x,t) given by (A.1.24), straightforward substitution of $\psi(x,k,t)$ in (A.1.1b) and taking limit $x \to \infty$ will not work. Instead of that, however, we can give a partial differential equation for b(k,t), with characteristics given by the solution $k(\kappa,t)$ of:

$$\text{(A.1.27)}\quad \begin{cases} \dfrac{dk^2}{dt} = f(k^2) \ , \\ k(0) = \kappa \ . \end{cases}$$

Of course, in order to be able to use inverse scattering, $k(\kappa,t)$ must cover the whole real axis. Therefore, we must restrict ourselves to time-intervals T, for which we have:

(A.1.28) $\forall t \in T$, $\exists$ interval $(m(t),M(t))$ so that $\varphi\colon (m(t),M(t)) \to \mathbb{R}$, defined by: $\varphi(\kappa) = k(\kappa,t)$, is a bijection.

Now substituting $\psi(x,k(\kappa,t),t)$ and (A.1.24) into (A.1.1b) and taking limit $x \to \infty$ leads to:

$$\text{(A.1.29)}\quad \frac{d}{dt}\, b(k(\kappa,t),t) = 2ik\, b_0^{(\lambda)}(t)\, b(k(\kappa,t),t) \ .$$

As a final step, we substitute (A.1.24) into (A.1.3b) to get an evolution equation for $u(x,t)$. We find:

$$\text{(A.1.30)}\quad u_t = \tfrac{1}{2}\left(N^{(\lambda)}u\right)_{xxx} + \frac{u}{\lambda}\, f(\lambda) + 2(\lambda + u)\left(N^{(\lambda)}u\right)_x + \left(b_0^{(\lambda)} + \frac{x}{2\lambda}\, f + N^{(\lambda)}u\right)u_x \ .$$

<u>Conclusion</u>:

Choose $b_0^{(\lambda)}(t)$ and $N^{(\lambda)}u$ so that:

1°. the right-hand side of (A.1.30) is λ-independent;

2°. $\lim\limits_{x\to\infty} (N^{\lambda}u)(x) = 0$.

Let T be a time interval satisfying (A.1.28). Then: (A.1.30) is S-integrable. The evolution of the spectral data is given by (A.1.20,26,29). Due to the fact that $b(\kappa,0) \equiv 0$ implies $b(k,t) \equiv 0$, these evolution equations have pure N-soliton solutions.

Examples of evolution equations of structure (A.1.30) are given by (2.1.17). For equation (2.1.17a) we see that Condition (A.1.28) is satisfied for all $t \in \mathbb{R}$.

The solitary wave solutions are given by:

$$\text{(A.1.31)} \quad 2\kappa^2 e^{pt} \operatorname{sech}^2\left(\kappa e^{\frac{1}{2}pt} x - \frac{8}{3p}\kappa^3\left(e^{\frac{3}{2}pt} - 1\right)\right) .$$

For equation (2.1.17b) we see that Condition (A.1.28) is satisfied for $p > 0$, $t \geqq 0$ and $p < 0$, $t \leqq 0$, respectively.
The solitary wave solutions are given by:

$$\text{(A.1.32)} \quad \frac{2\kappa^2}{1 + 2p\kappa^2 t} \operatorname{sech}^2\left(\frac{\kappa x}{\sqrt{1+2p\kappa^2 t}} - \frac{4\kappa}{p}\left(1 - \frac{1}{\sqrt{1+2p\kappa^2 t}}\right) + \tfrac{1}{2}\log(1 + 2p\kappa^2 t)\right) .$$

A.2. Proofs of Theorem (2.2.3), (2.2.36), Theorem (2.2.4), Theorem (2.2.6) and Lemma (2.2.1)

i) Proof of Theorem (2.2.3)

First, we note that with Theorem (2.2.1) and (2.2.5), it is easy to see that $u \in C^m(\mathbb{R})$ and $u = [0]$, implies that

$$R(x,k) \in C^{m+2}(\mathbb{R}) \quad \text{for all values of } k \in \bar{\mathbb{C}}_+ \setminus \{0\} .$$

We prove (2.2.31) with induction to p.
For $p = 0$, we have:

$$R'(x,k) = G(x,x,k)R(x,k) + \int_{-\infty}^{\infty} u(y) e^{2ik(x-y)} R(y,k)\,dy =$$

$$= \int_{-\infty}^{x} u(y) e^{2ik(x-y)} R(y,k)\,dy .$$

The induction step is proved by:

$$R^{(p_0+1)}(x,k) = \frac{d}{dx}\left\{\int_{-\infty}^{x} e^{2ik(x-y)} \frac{d^{p_0-1}}{dy^{p_0-1}}(uR)\,dy\right\} =$$

$$= \frac{d^{p_0-1}}{dx^{p_0-1}}(uR) + 2ik \int_{-\infty}^{x} e^{2ik(x-y)} \frac{d^{p_0-1}}{dy^{p_0-1}}(uR)\,dy =$$

$$= \frac{d^{p_0-1}}{dx^{p_0-1}}(uR) - \left[e^{2ik(x-y)}\frac{d^{p_0-1}}{dy^{p_0-1}}(uR)\right]_{y=-\infty}^{y=x} +$$

$$+ \int_{-\infty}^{x} e^{2ik(x-y)}\frac{d^{p_0}}{dy^{p_0}}(uR)dy =$$

$$= \int_{-\infty}^{x} e^{2ik(x-y)}\frac{d^{p_0}}{dy^{p_0}}(uR)dy \ .$$

In the last step, we have used that it is already known that (2.2.31) is fulfilled for $p \leqq p_0$, so that:

$$\lim_{x\to-\infty} R^{(p)}(x,k) = 0 \ , \quad 1 \leqq p \leqq p_0 \ .$$

(2.2.32a) is a trivial consequence of (2.2.30) and (2.2.31).
(2.2.32b) can be proved by induction too, using the relationship:

$$R^{(p+1)}(x,k) = \int_{-\infty}^{x} e^{2ik(x-y)}\frac{d^p}{dy^p}(uR)dy =$$

$$= -\frac{1}{2ik}\frac{d^p}{dx^p}(uR) + \frac{1}{2ik}\int_{-\infty}^{x} e^{2ik(x-y)}\frac{d^{p+1}}{dy^{p+1}}(uR)dy \ .$$

Q.E.D.

ii) Proof of (2.2.36)

Theorem (2.2.1c) implies:

i) $\tilde{W}(k) \in C^1(\mathbb{R}) \Rightarrow r_+(k) = \frac{1}{2ik}\left(\tilde{W}(0) + k\frac{d\tilde{W}}{dk}(0)\right) + \tilde{w}_r(k)$, with $\tilde{w}_r(k) \in C(\mathbb{R})$ and $\lim_{k\to0}\tilde{w}_r(k) = 0$.

ii) $W(k) \in C^1(\bar{\mathbb{C}}_+) \Rightarrow r_-(k) = \frac{1}{2ik}\left(W(0) + k\frac{dW}{dk}(0)\right) + w_r(k)$, with $w_r(k) \in C(\bar{\mathbb{C}}_+)$ and $\lim_{k\to0} w_r(k) = 0$.

From (2.2.22), we know that: $|r_-(k)| \geq 1$, $k \in \mathbb{R}$. So:

$$\text{Either } W(0) \neq 0, \text{ or } W(0) = 0 \text{ and } \left|\frac{dW}{dk}(0)\right| \geq 2 .$$

We also have:

$$\tilde{W}(0) = - W(0) .$$

Combining the above results leads to:
If $W(0) = 0$, then:

$$\frac{dW}{dk}(0) \neq 0 \; ; \quad a(0) = \frac{1}{r_-(0)} = 2i \left(\frac{dW}{dk}(0)\right)^{-1} \; ;$$

$$b(0) = \frac{r_+(0)}{r_-(0)} = \left(\frac{d\tilde{W}}{dk}(0)\right)\left(\frac{dW}{dk}(0)\right)^{-1} .$$

If $W(0) \neq 0$, then:

$$a(k) \sim \frac{2ik}{W(0)} , \quad k \to 0 \; ; \quad b(0) = -1 .$$

Q.E.D.

iii) Proof of Theorem (2.2.4)

First, we prove that:
If:

(A.2.1) $\quad u^{(p)}(x)$ is bounded for $x \to -\infty$ and $u^{(p)}(x) = [0]$, $0 \leq p \leq m$,

then:

$$\text{(A.2.2)} \quad \frac{\partial^p}{\partial x^p} G_n(x,k) = 0\left(\frac{1}{|k|^n}\right) , \quad |k| \to \infty , \; k \in \bar{\mathbb{C}}_+ , \; 0 \leq p \leq m .$$

We have the following relationship between $G_{n+1}^{(p)}(x,k)$ and $G_n^{(p)}(x,k)$:

$$\text{(A.2.3)} \quad G_{n+1}^{(p)} = \frac{\partial^p}{\partial x^p}\left(\int_{-\infty}^{x} G(x,y,k)G_n(y,k)dy\right) =$$

$$= \frac{\partial^{p-1}}{\partial x^{p-1}}\left(\int_{-\infty}^{x} e^{2ik(x-y)}\, u(y)\, G_n(y,k)dy\right) =$$

$$= \frac{\partial^{p-2}}{\partial x^{p-2}} \left(uG_n + 2ik \int_{-\infty}^{x} e^{2ik(x-y)} \, uG_n \, dy \right) =$$

$$= \frac{\partial^{p-2}}{\partial x^{p-2}} \left(\int_{-\infty}^{x} e^{2ik(x-y)} (uG_n)^{3} \, dy \right) = \ldots\ldots =$$

$$= \int_{-\infty}^{x} e^{2ik(x-y)} (uG_n)^{p-1} \, dy \;, \quad n \geqq 0 \;, \quad p \geqq 1 \;.$$

Note that in (A.2.3), we have used that

$$\text{(A.2.4)} \qquad \lim_{x \to -\infty} G_n^{(p)}(x,k) = 0 \;, \quad n \geqq 1 \;.$$

The validity of (A.2.3) for $n = 0$ follows with conditions (A.2.1). So, (A.2.4) holds for $n = 1$. By induction, the validity of (A.2.3) and (A.2.4) follows for all n.

Using (A.2.3) for $n = 0$, we find:

$$G_1^{(p)} = \int_{-\infty}^{x} e^{2ik(x-y)} \, u^{(p-1)}(y) dy =$$

$$= - \frac{1}{2ik} \, u^{(p-1)}(x) + \frac{1}{2ik} \int_{-\infty}^{x} e^{2ik(x-y)} \, u^{(p)}(y) dy \;.$$

So, (A.2.2) holds for $n = 1$. The validity of (A.2.2) for all n is again proved by induction to n using the relationship:

$$G_{n+1}^{(p)} = \int_{-\infty}^{x} e^{2ik(x-y)} (uG_n)^{p-1} \, dy =$$

$$= - \frac{1}{2ik} (uG_n)^{p-1} + \frac{1}{2ik} \int_{-\infty}^{x} e^{2ik(x-y)} (uG_n)^{(p)} \, dy \;.$$

We will now prove the actual theorem.

<u>ad a)</u>:

With (2.2.15), (2.2.25,26) and (A.2.2), a straightforward calculation leads to:

$$r_-(k) = 1 - \frac{1}{2ik}\int_{-\infty}^{\infty} u(x)dx + \frac{1}{2}\frac{1}{(2ik)^2}\left(\int_{-\infty}^{\infty} u(x)dx\right)^2 +$$

$$- \frac{1}{6}\frac{1}{(2ik)^2}\left(\int_{-\infty}^{\infty} u(x)dx\right)^3 + \frac{1}{(2ik)^3}\int_{-\infty}^{\infty} u^2(x)dx + O\left(\frac{1}{|k|^4}\right),$$

$$|k| \to \infty, \quad k \in \bar{\mathbb{C}}_+ .$$

Taking $k = \xi + i\eta$ and using (2.2.21), we now find:

$$|r_-(k)|^2 = r_-(k)r_-(-\bar{k}) = 1 - \frac{\eta}{|k|^2} U_0 + O\left(\frac{1}{|k|^3}\right), \quad |k| \to \infty,$$

$$|r_-(k)|^2 = 1 + \frac{1}{4}\frac{\eta(3\xi^2 - \eta^2)}{|k|^6}\int_{-\infty}^{\infty} u^2(x)dx + \ldots\ldots =$$

$$= 1 + O\left(\frac{1}{|k|^4}\right), \quad |k| \to \infty, \quad \text{if } U_0 = 0 .$$

ad b):

Consider (2.2.16).

m-times partial integration and using $\lim_{x\to-\infty} R^{(p)}(x,k) = 0$, $p \geqq 1$, leads to:

$$r_+(k) = \frac{1}{(2ik)^{m+1}}\int_{-\infty}^{\infty} e^{-2iky}(uR)^{(m)}\, dy, \quad k \in \mathbb{R}\setminus\{0\} .$$

So:

$$|b(k)|^2 = b(k)b(-k) = \frac{r_+(k)r_+(-k)}{r_-(k)r_-(-k)} = \frac{O\left(|k|^{-2(m+1)}\right)}{O\left(1 + \frac{1}{|k|}\right)} =$$

$$= O\left(|k|^{-2(m+1)}\right), \quad |k| \to \infty, \quad k \in \mathbb{R},$$

and

$$1 - |a(k)| = \frac{1 - |a(k)|^2}{1 + |a(k)|} = \frac{|b(k)|^2}{1 + |a(k)|} = O\left(|k|^{-2(m+1)}\right),$$

$$|k| \to \infty, \quad k \in \mathbb{R}.$$

Q.E.D.

iv) Proof of Theorem (2.2.6)

Consider (2.2.50). (Because u = [2], this expression is well-defined.) Taking the logarithm leads to:

$$\log r_-(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log(1-|b(q)|^2)}{k-q}\, dq + \sum_{\ell=1}^{N} \log \frac{k-ik_\ell}{k+ik_\ell}, \quad \mathrm{Im}\, k > 0 .$$

Using

$$\frac{1}{k-q} = \frac{1}{k} \sum_{n=0}^{p} \left(\frac{q}{k}\right)^n + \frac{1}{k-q}\left(\frac{q}{k}\right)^{p+1}$$

and

$$\log \frac{k-ik_\ell}{k+ik_\ell} = -2 \sum_{n=0}^{\infty} \left(\frac{ik_\ell}{k}\right)^{2n+1} \cdot \frac{1}{2n+1}, \quad \left|\frac{k_\ell}{k}\right| < 1 ,$$

we can expand $\log r_-(k)$ for $|k| \to \infty$ in powers of $1/k$.

(A.2.5) $$\log r_-(k) = \sum_{n=0}^{p+1} \frac{\alpha_n}{k^n} + \frac{1}{k^{p+1}} \int_{-\infty}^{\infty} \frac{q^{p+1} \log(1-|b(q)|^2)}{k-q}\, dq + {}$$

$$+ O\left(|k|^{-(p+2)}\right), \quad |k| \to \infty, \quad \mathrm{Im}\, k > 0 ,$$

with

(A.2.6) $$\begin{cases} \alpha_{2n} = 0 \quad (\text{since } |b(q)|^2 = b(q)b(-q) \text{ is even}), \\ \alpha_{2n+1} = \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} q^{2n} \log(1-|b(q)|^2)dq - \frac{2}{2n+1} \sum_{\ell=1}^{N} (ik_\ell)^{2n+1} . \end{cases}$$

Using Theorem (2.2.4b), we see that under the conditions of Theorem (2.2.6) the expansion (A.2.5) is well-defined for $0 \leqq p \leqq 2m$, since the integrals converge.

We will now derive an expansion of $\log r_-(k)$ in another way. We define:

(A.2.7) $$\sigma(x,k) = \frac{R'(x,k)}{R(x,k)}, \quad |k| \text{ large}, \quad \mathrm{Im}\, k > 0, \quad r_-(k) \neq 0 .$$

Since $|R(x,k)| = 1 + O(1/|k|$, $|k| \to \infty$, is positive for $|k|$ large, $\sigma(x,k)$ is well-defined.

With (2.2.5b) and (2.2.34) we see that:

$$\text{(A.2.8)} \qquad \int_{-\infty}^{\infty} \sigma(x,k)dx = \log R(x,k)\Big]_{-\infty}^{\infty} = \log r_-(k) \ .$$

By using (2.2.5a), it follows that σ solves the following differential equation:

$$\text{(A.2.9)} \qquad \sigma_x + \sigma^2 - u - 2ik\sigma = 0 \ .$$

From Theorem (2.2.2) and (2.2.5a), we know that for $|k| \to \infty$ we can expand σ, as well as σ_x, in powers of $1/k$, in such a way that the first p terms approximate σ with order $|k|^{-(p+1)}$ uniformly in x on $\mathbb{R}$.
We search for a solution of (A.2.9) by substitution of a power series:

$$\text{(A.2.10)} \qquad \sigma(x,k) = \sum_{n=1}^{p+1} \frac{\sigma_n(x)}{(2ik)^n} + \tilde{\sigma}_{p+2}(x,k) \ ,$$

where $\sigma_n(x)$ and $\frac{d}{dx}\sigma_n(x)$ are bounded, and $\tilde{\sigma}_{p+2}(x,k)$ and $\frac{\partial}{\partial x}\tilde{\sigma}_{p+2}(x,k)$ are $O(1/|k|^{p+2})$ uniformly in x on $\mathbb{R}$.
This gives:

$$\text{(A.2.11)} \qquad \sum_{n=1}^{p} \frac{(\sigma_n)_x}{(2ik)^n} + \sum_{n=2}^{p} \frac{\left(\sum_{i+j=n} \sigma_i\sigma_j\right)}{(2ik)^n} - \sum_{n=0}^{p} \frac{\sigma_{n+1}}{(2ik)^n} - u = O\left(|k|^{-(p+1)}\right) \ .$$

So we find:

$$\text{(A.2.12)} \qquad \sigma_1(x) = -u(x) \ ; \quad \sigma_2(x) = -\frac{du}{dx}(x) \ ;$$

$$\sigma_{n+1} = (\sigma_n)_x + \sum_{i+j=n} \sigma_i\sigma_j \ , \qquad 2 \leq n \leq p \ .$$

We see that σ_n has the structure of a polynomial in u and x-derivatives, up to order $n-1$, of u. Therefore, under the conditions of the theorem, $\int_{-\infty}^{\infty} \sigma_n(x)dx$ is well-defined for $1 \leq n \leq m+1$.

Combining the above facts, we find that:

$$\text{(A.2.13)} \qquad \log r_-(k) = \sum_{n=1}^{m+1} \frac{\alpha_n}{k^n} + O\left(|k|^{-m-2}\right) \ , \quad |k| \to \infty \ , \quad \text{Im } k > 0 \ ,$$

where

(A.2.14) $\alpha_n = \frac{1}{(2i)^n} \int_{-\infty}^{\infty} \sigma_n(x)dx$, σ_n defined by (A.2.12) .

Comparing the expansions (A.2.5,6) and (A.2.13,14) proves the theorem. (That indeed $\alpha_{2n} = 0$ in the expression (A.2.14) can also be seen from the fact that $\sigma_{2n}(x)$ is a total derivative, e.g.: $\sigma_2 = -u_x$, $\sigma_4 = -u_{xxx} + 4uu_x = \frac{d}{dx}(-u_{xx} + 2u^2)$.)

Q.E.D.

v) Proof of Lemma (2.2.1)

From elementary linear algebra, we know that:

$$\det(I + C) = 1 + \det C + \sum_{m=1}^{N-1} C_m ,$$

where C_m is the sum of the determinants of all $m \times m$ diagonal submatrices of C.

C, as well as each $m \times m$ diagonal submatrix of C, is of the form B, where B is given by:

$$B = \left(\frac{c_{n_i} c_{n_\ell}}{k_{n_i} + k_{n_\ell}} e^{-(k_{n_i}+k_{n_\ell})x} \right)_{i,\ell = 1,\dots,j} ,$$

where $n_1 < n_2 < \dots < n_j$ is some subsequence of $\{1,\dots,N\}$.

So, B is a symmetric matrix. Moreover, B is positive definite since it holds that

$$\xi^T B \xi = \sum_{i,\ell=1}^{j} c_{n_i} c_{n_\ell} \xi_{n_i} \xi_{n_\ell} \frac{e^{-(k_{n_i}+k_{n_\ell})x}}{k_{n_i} + k_{n_\ell}} =$$

$$= \int_x^\infty \sum_{i,\ell=1}^{j} c_{n_i} c_{n_\ell} \xi_{n_i} \xi_{n_\ell} e^{-(k_{n_i}+k_{n_\ell})z} dz =$$

$$= \int_x^\infty \left(\sum_{i=1}^{j} c_{n_i} \xi_{n_i} e^{-k_{n_i} z} \right)^2 dz > 0 , \quad \forall \xi = (\xi_{n_1},\dots,\xi_{n_j})^T \neq 0 .$$

Since the determinant of a real symmetric positive definite matrix is positive, the lemma has now been proved.

Q.E.D.

APPENDIX B

B.1. The evolution equations for $\gamma_n(t)$ and a(k,t)

Starting point for the derivation of the evolution equations for the spectral data, is the following evolution equation for a (generalized) eigenfunction $\psi_k(x,t)$ corresponding to $\lambda = k^2$, $k \in \bar{\mathbb{C}}_+$:

(B.1.1) $$\left[\frac{d^2}{dx^2} - u(x,t) + k^2\right]\left(\frac{\partial}{\partial t}\psi_k - B\psi_k\right) = (\varepsilon f(u) - \lambda_t)\psi_k ,$$

$$B\psi_k = -2(u+2k^2)\frac{\partial}{\partial x}\psi_k + (u_x - c)\psi_k ,$$

where c is an undetermined constant.
The derivation of this equation can be found in [EvH], § 7.1. As in [EvH], we define:

(B.1.2) $$R = \frac{\partial}{\partial t}\psi_k - B\psi_k .$$

Now, multiplying (B.1.1) by an arbitrary eigenfunction $\hat{\psi}_k$ corresponding to $\lambda = k^2$, followed by integration between any two points x and x_0, we obtain:

(B.1.3) $$\left[\hat{\psi}_k\frac{\partial R}{\partial x'} - R\frac{\partial}{\partial x'}\hat{\psi}_k\right]_{x'=x} - \left[\hat{\psi}_k\frac{\partial R}{\partial x'} - R\frac{\partial}{\partial x'}\hat{\psi}_k\right]_{x'=x_0} =$$

$$= \int_{x_0}^{x}(\varepsilon f(u) - \lambda_t)\psi_k\hat{\psi}_k\,dx' .$$

All the evolution equations for the spectral data can be found by putting into (B.1.3) various choices for the (generalized) eigenfunctions ψ_k and $\hat{\psi}_k$ and using their asymptotic behaviour for $|x| \to \infty$.

For the evolution equation of $\gamma_n(t)$, we take:

(B.1.4) $$\lambda = \lambda_n = -k_n^2, \quad \psi_k = \tilde{\psi}_n, \quad \hat{\psi}_k = \tilde{\phi}_n .$$

We remember that the asymptotic behaviour of $\tilde{\psi}_n$ and $\tilde{\phi}_n$ is given by:

$$\text{(B.1.5)}\qquad \lim_{x\to\infty} \tilde{\psi}_n e^{k_n x} = c_n \;; \qquad \lim_{x\to\infty} \tilde{\phi}_n \tilde{\psi}_n = -1 \;;$$

$$\lim_{x\to-\infty} \tilde{\psi}_n e^{-k_n x} = 1 \;; \qquad \lim_{x\to-\infty} \tilde{\phi}_n \tilde{\psi}_n = 1 \;.$$

Moreover, we have the following relationship between $\tilde{\phi}_n$ and $\tilde{\psi}_n$:

$$\text{(B.1.6)}\qquad W(\tilde{\phi}_n,\tilde{\psi}_n) = \tilde{\phi}_n \frac{\partial}{\partial x}\tilde{\psi}_n - \tilde{\psi}_n \frac{\partial}{\partial x}\tilde{\phi}_n = 2k_n \;.$$

Substituting (B.1.4) in (B.1.3) and taking the limit $x_0 \to -\infty$ gives:

$$\text{(B.1.7)}\qquad \left[\tilde{\phi}_n \frac{\partial R}{\partial x'} - R\frac{\partial}{\partial x'}\tilde{\phi}_n\right]_{x'=x} = \varepsilon \int_{-\infty}^{x} f(u)\tilde{\phi}_n\tilde{\psi}_n \, dx' +$$

$$-\frac{d\lambda_n}{dt}\left\{\int_{-\infty}^{x} (\tilde{\phi}_n\tilde{\psi}_n - 1)dx' + x\right\} + 8k_n^4 + \frac{d}{dt}k_n - 2ck_n =: F(x,t) \;.$$

Using

$$\frac{\partial}{\partial x}\frac{R}{\tilde{\Phi}_n} = \frac{1}{\tilde{\Phi}_n^2}\left(\frac{\partial R}{\partial x}\tilde{\Phi}_n - R\frac{\partial\tilde{\Phi}_n}{\partial x}\right) \quad\text{and}\quad \frac{\partial}{\partial x}\frac{\tilde{\psi}_n}{\tilde{\Phi}_n} = \frac{2k_n}{\tilde{\Phi}_n^2} \,,$$

integration of (B.1.7) gives:

$$\frac{R(x)}{\tilde{\Phi}_n(x)} - \frac{R(x_0)}{\tilde{\Phi}_n(x_0)} = \int_{x_0}^{x} \frac{F}{\tilde{\Phi}_n^2}\, dx' = \left[\frac{1}{2k_n}\frac{\tilde{\psi}_n}{\tilde{\Phi}_n}F\right]_{x_0}^{x} - \frac{1}{2k_n}\int_{x_0}^{x} \frac{\tilde{\psi}_n}{\tilde{\Phi}_n}\frac{\partial}{\partial x'}F\, dx' \,.$$

Taking the limit for $x_0 \to -\infty$, using (B.1.5), we arrive at:

$$R = \frac{1}{2k_n}\tilde{\psi}_n F - \frac{\varepsilon\tilde{\Phi}_n}{2k_n}\int_{-\infty}^{x} f\,\tilde{\psi}_n^2\, dx' + \frac{\varepsilon\tilde{\Phi}_n}{2\gamma_n k_n}\left(\int_{-\infty}^{\infty} f\,\tilde{\psi}_n^2\, dx\right)\left(\int_{-\infty}^{x}\tilde{\psi}_n^2\, dx'\right) \,.$$

Using (B.1.2,4,7), this transforms to:

$$\text{(B.1.8)}\qquad \tilde{\psi}_n \frac{\partial\tilde{\psi}_n}{\partial t} = 2(u-2k_n^2)\frac{\partial\tilde{\psi}_n}{\partial x}\tilde{\psi}_n - \left(\frac{\partial u}{\partial x} - c\right)\tilde{\psi}_n^2 +$$

$$+ \frac{\varepsilon}{2k_n} \tilde{\psi}_n \Big\{ \tilde{\psi}_n \int_{-\infty}^{x} f \tilde{\Phi}_n \tilde{\psi}_n \, dx' - \frac{\tilde{\psi}_n}{\gamma_n} \Big(\int_{-\infty}^{\infty} f \tilde{\psi}_n^2 \, dx \Big) \cdot$$

$$\cdot \Big(\int_{-\infty}^{x} (\tilde{\Phi}_n \tilde{\psi}_n - 1) dx' + x + \frac{1}{2k_n} \Big) - \tilde{\Phi}_n \int_{-\infty}^{x} f \tilde{\psi}_n^2 \, dx' +$$

$$+ \frac{\tilde{\Phi}_n}{\gamma_n} \Big(\int_{-\infty}^{\infty} f \tilde{\psi}_n^2 \, dx \Big) \Big(\int_{-\infty}^{x} \tilde{\psi}_n^2 \, dx' \Big) \Big\} - c \tilde{\psi}_n^2 + 4k_n^3 \tilde{\psi}_n^2 \, .$$

Integrating (B.1.8) over $\mathbb{R}$ and using (3.1.8,9) and

$$- 2u \tilde{\psi}_n \frac{\partial \tilde{\psi}_n}{\partial x} + \frac{\partial u}{\partial x} \tilde{\psi}_n^2 = \frac{\partial}{\partial x} (u \tilde{\psi}_n^2) - 4u \tilde{\psi}_n \frac{\partial \tilde{\psi}_n}{\partial x} = \frac{\partial}{\partial x} (u \tilde{\psi}_n^2) +$$

$$- 4 \Big(\frac{\partial^2 \psi_n}{\partial x^2} - k_n^2 \tilde{\psi}_n \Big) \frac{\partial \tilde{\psi}_n}{\partial x} = \frac{\partial}{\partial x} \Big((u + 2k_n^2) \tilde{\psi}_n^2 - 2 \Big(\frac{\partial \tilde{\psi}_n}{\partial x} \Big)^2 \Big) \, ,$$

results in

$$\frac{\partial}{\partial t} \gamma_n(t) - 8k_n^3(t) \gamma_n(t) = \frac{\varepsilon}{k_n(t)} \gamma_n(t) G_n(t) \, ,$$

with

$$G_n := \int_{-\infty}^{\infty} \phi_n \psi_n \Big\{ \Big(\int_{-\infty}^{\infty} f \psi_n^2 \, dx \Big) \int_{-\infty}^{x} \psi_n^2 \, dx' - \int_{-\infty}^{x} f \psi_n^2 \, dx' \Big\} dx +$$

$$+ \int_{-\infty}^{\infty} \psi_n^2 \Big(\int_{-\infty}^{x} f \phi_n \psi_n \, dx' \Big) dx +$$

$$- \Big(\int_{-\infty}^{\infty} f \psi_n^2 \, dx \Big) \int_{-\infty}^{\infty} \psi_n^2 \Big\{ x + \int_{-\infty}^{x} (\phi_n \psi_n - 1) dx' + \frac{1}{2k_n} \Big\} dx \, .$$

This corresponds to the equation for $\gamma_n(t)$ given in (3.1.10).

For the evolution equation of $a(k,t)$ we take $\lambda = k^2$, $k \in \mathbb{R}$, arbitrary but fixed. So $\lambda_t = 0$. The generalized eigenfunctions are chosen as follows:

(B.1.9) $\quad \hat{\psi}_k = \psi_k = a\psi_r + a\bar{\psi}_r \, , \quad \psi_r$ as defined in (2.2.4) .

The asymptotic behaviour of ψ_k is given by:

$$\psi_k \sim a e^{-ikx} + \bar{a} e^{ikx} \quad , \quad x \to -\infty \ ,$$

$$\psi_k \sim (1+b) e^{ikx} + (1+\bar{b}) e^{-ikx} \quad , \quad x \to \infty \ .$$

This leads to:

$$\psi_k \frac{\partial R}{\partial x} - R \frac{\partial}{\partial x} \psi_k \sim 2ik(a\bar{a}_t - \bar{a} a_t) + 16k^4 a\bar{a} \quad , \quad x \to -\infty \ ,$$

$$\psi_k \frac{\partial R}{\partial x} - R \frac{\partial}{\partial x} \psi_k \sim 2ik\big((1+\bar{b})b_t - (1+b)\bar{b}_t\big) + 16k^4 (1+b)(1+\bar{b}) \ , \quad x \to \infty \ .$$

Introducing al this information into (B.1.3) and taking the limits $x_0 \to -\infty$ and $x \to \infty$, we get:

(B.1.10) $$2ik \frac{d}{dt}(b - \bar{b}) + 2ik(\bar{b} b_t - b\bar{b}_t) + 16k^4 (b + \bar{b} + 2b\bar{b}) + 2ik\bar{a}^{-2} \frac{d}{dt} \frac{a}{\bar{a}} =$$

$$= \varepsilon \int_{-\infty}^{\infty} f(u) \psi_k^2 \, dx \ .$$

(We have used that: $|a|^2 + |b|^2 = 1$.)

From the evolution equation (3.1.5) for $b(k,t)$ and $\psi(x,k,t) = a(k,t)\psi_r(x,k,t)$, we see that:

(B.1.11) $$2ik \frac{d}{dt}(b - \bar{b}) + 2ik(\bar{b} b_t - b\bar{b}_t) + 16k^4 (b + \bar{b} + 2b\bar{b}) =$$

$$= \varepsilon(1+\bar{b})a^2 \int_{-\infty}^{\infty} f(u)\psi_r^2 \, dx + \varepsilon(1+b)\bar{a}^2 \int_{-\infty}^{\infty} f(u)\bar{\psi}_r^2 \, dx \ .$$

Combining (B.1.9,10,11), we arrive at:

(B.1.12) $$2\varepsilon a\bar{a} \int_{-\infty}^{\infty} f(u)\psi_r \bar{\psi}_r \, dx =$$

$$= \varepsilon \bar{b} a^2 \int_{-\infty}^{\infty} f(u)\psi_r^2 \, dx + \varepsilon b\bar{a}^2 \int_{-\infty}^{\infty} f(u)\bar{\psi}_r^2 \, dx + 2ik\bar{a}^2 \frac{d}{dt} \frac{a}{\bar{a}} \ .$$

We now write:

$$a(k,t) = |a(k,t)|\, e^{i\varphi_a(k,t)} .$$

By dividing (B.1.12) by $\bar{a}^2$, we find the following evolution equation for the phase of the transmission coefficient $\varphi_a(k,t)$:

$$\text{(B.1.13)} \quad 4k \frac{\partial}{\partial t} \varphi_a = \varepsilon \Big\{ -2 \int_{-\infty}^{\infty} f(u)|\psi_r|^2 dx + \bar{b} e^{2i\varphi_a} \int_{-\infty}^{\infty} f(u)\psi_r^2 dx + + b e^{-2i\varphi_a} \int_{-\infty}^{\infty} f \bar{\psi}_r^2 dx \Big\} .$$

The modulus $|a(k,t)|$ of the transmission coefficient is determined by:

$$\text{(B.1.14)} \quad |a(k,t)|^2 = 1 - |b(k,t)|^2 .$$

So the evolution of the transmission coefficient is given by the equations (3.1.5), (B.1.13) and (B.1.14).

B.2. Well-definedness of θ_n and H_n

In the following lemmas, ε and t are considered to be parameters. The constants in the proofs are generic, i.e. they have different values in different parts of the proofs. We take ε and t arbitrary but fixed.

Lemma (B.2.1):
If u(x) *satisfies a growth-condition of order* 1, *then,*

$$\theta_n = \lim_{x\to\infty} \Big\{ \int_{-\infty}^{x} (\tilde{\phi}_n \tilde{\psi}_n - 1)dx' + x \Big\}$$

exists.

Proof:
From (2.2.41,23), we know that $\tilde{\psi}_n(x)$ can be written as:

$$\text{(B.2.1)} \quad \tilde{\psi}_n(x) = R(x, ik_n)e^{k_n x} ,$$

where R(x,k) satisfies:

(B.2.2) $$R(x,k) = 1 + \int_{-\infty}^{x} \frac{u(y)}{2ik} \{e^{2ik(x-y)} - 1\} R(y,k)dy \ , \quad k \in \bar{\mathbb{C}}_+ \setminus \{0\} \ ,$$

and

(B.2.3) $$\lim_{x \to -\infty} R(x,k) = 1 \ ; \ R(x,k) \text{ is continuous in } (x,k) \text{ on } \mathbb{R} \times \bar{\mathbb{C}}_+ \ .$$

It also holds that:

$$\theta_n = \lim_{x\to\infty} \left\{ \int_{-\infty}^{x} (\tilde{\phi}_n \tilde{\psi}_n - 1)dx' + 2x \right\} = \int_{-\infty}^{0} (\tilde{\phi}_n \tilde{\psi}_n - 1)dx + \int_{0}^{\infty} (\tilde{\phi}_n \tilde{\psi}_n + 1)dx \ ,$$

provided that both integrals on the right-hand side converge.
We will now show convergence of $\int_{-\infty}^{0} (\tilde{\phi}_n \tilde{\psi}_n - 1)dx$. From (B.2.1.3) we see that

$$\exists L \text{ such that } \tilde{\psi}_n(x) > 0 \quad \text{for } x \leq L \ .$$

It can be easily verified that, for $x \leq L$, the solution $\tilde{\phi}_n(x)$ of the S.E. can be represented by:

$$\tilde{\phi}_n(x) = \left[\frac{\tilde{\phi}_n(L)}{\tilde{\psi}_n(L)} + 2k_n \int_{x}^{L} \tilde{\psi}_n^{-2}(\xi)d\xi \right] \tilde{\psi}_n(x) \ , \quad x \leq L \ .$$

We now have:

(B.2.4) $$\int_{-\infty}^{0} (\tilde{\phi}_n \tilde{\psi}_n - 1)dx = \int_{L}^{0} (\tilde{\phi}_n \tilde{\psi}_n - 1)dx + $$
$$+ 2k_n \int_{-\infty}^{L} \left\{ \tilde{\psi}_n^2(x) \int_{x}^{L} \tilde{\psi}_n^{-2}(\xi)d\xi - \frac{1}{2k_n} \right\} dx + \frac{\tilde{\phi}_n(L)}{\tilde{\psi}_n(L)} \int_{-\infty}^{L} \tilde{\psi}_n^2(x)dx \ .$$

From this equation, we see that $\int_{-\infty}^{0} (\tilde{\phi}_n \tilde{\psi}_n - 1)dx$ converges, iff

(B.2.5) $$\int_{-\infty}^{L} \left\{ \tilde{\psi}_n^2(x) \int_{x}^{L} \tilde{\psi}_n^{-2}(\xi)d\xi - \frac{1}{2k_n} \right\} dx$$

is convergent.

From (B.2.1,3) it trivially follows that:

$$\text{(B.2.6)} \qquad 1 - f_n(x) \leq \tilde{\psi}_n(x)e^{-k_n x} \leq 1 + f_n(x) \ ,$$

where $f_n(x) := |1 - R(x, ik_n)|$ is a positive continuous function which tends to zero for $x \to -\infty$.

With (B.2.6) it follows that:

$$\text{(B.2.7)} \qquad \left| \int_{-\infty}^{L} \left\{ \tilde{\psi}_n^2(x) \int_x^L \tilde{\psi}_n^{-2}(\xi)\,d\xi - \frac{1}{2k_n} \right\} dx \right| \leq$$

$$\leq \int_{-\infty}^{L} \left| \left(1 + f_n(x)\right)^2 e^{2k_n x} \int_x^L \frac{e^{-2k_n \xi}}{\left(1 - f_n(\xi)\right)^2}\,d\xi - \frac{1}{2k_n} \right| dx \leq$$

$$\leq \int_{-\infty}^{L} \left| e^{2k_n x} \int_x^L e^{-2k_n \xi}\,d\xi - \frac{1}{2k_n} \right| dx \ +$$

$$+ \int_{-\infty}^{L} e^{2k_n x} \int_x^L e^{-2k_n \xi} \left\{ -1 + \frac{1}{\left(1 - f_n(\xi)\right)^2} \right\} d\xi\,dx \ +$$

$$+ \int_{-\infty}^{L} \left(2f_n(x) + f_n^2(x)\right) e^{2k_n x} \int_x^L \frac{e^{-2k_n \xi}}{\left(1 - f_n(\xi)\right)^2}\,d\xi\,dx =$$

$$=: I_1 + I_2 + I_3 \ ,$$

with

$$I_1 = \frac{1}{2k_n} \int_{-\infty}^{L} \left| e^{2k_n x} \left(- e^{-2k_n L} + e^{-2k_n x} \right) - 1 \right| dx =$$

$$= \int_{-\infty}^{L} \frac{e^{2k_n (x-L)}}{2k_n}\,dx = \frac{1}{4k_n^2} \ .$$

Using

$$\frac{1}{\left(1 - f_n(\xi)\right)^2} - 1 = \frac{2f_n(\xi) - f_n^2(\xi)}{\left(1 - f_n(\xi)\right)^2} \leq C \cdot f_n(\xi) \quad (\text{e.g. } C = 4 \text{ if } f_n(\xi) < \tfrac{1}{2}) \ ,$$

we get:

$$I_2 \leq C \int_{-\infty}^{L} e^{2k_n x} \int_{x}^{L} f_n(\xi) e^{-2k_n \xi} d\xi \, dx =$$

$$= C\left[\frac{e^{2k_n x}}{2k_n} \int_{x}^{L} f_n(\xi) e^{-2k_n \xi} d\xi\right]_{-\infty}^{L} + \frac{C}{2k_n} \int_{-\infty}^{L} f_n(x) dx \; ;$$

$$\lim_{x \to -\infty} e^{2k_n x} \int_{x}^{L} f_n(\xi) e^{-2k_n \xi} d\xi \leq C \lim_{x \to -\infty} e^{2k_n x} \int_{x}^{L} e^{-2k_n \xi} d\xi =$$

$$= C \lim_{x \to -\infty} \frac{e^{2k_n x}}{2k_n} \left(e^{-2k_n x} - e^{-2k_n L}\right) = \frac{C}{2k_n} < \infty .$$

So I_2 converges if $\int_{-\infty}^{L} f_n(x)dx$ converges.

$$I_3 \leq C \int_{-\infty}^{L} \left(2f_n(x) + f_n^2(x)\right) e^{2k_n x} \int_{x}^{L} e^{-2k_n \xi} d\xi \, dx \leq$$

$$\leq C \int_{-\infty}^{L} \left(2f_n(x) + f_n^2(x)\right) \left(1 - e^{2k_n (x-L)}\right) dx .$$

So, I_3 converges if $\int_{-\infty}^{L} f_n(x)dx$ converges.

This leaves us to prove the convergence of $\int_{-\infty}^{L} f_n(x)dx$. Using (B.2.2) we get:

$$\int_{-\infty}^{L} f_n(x) dx = \int_{-\infty}^{L} |1 - R(x, ik_n)| \, dx =$$

$$= \int_{-\infty}^{L} \frac{1}{2k_n} \left| \int_{-\infty}^{x} u(y) \left(1 - e^{-2k_n (x-y)}\right) R(y, ik_n) dy \right| dx \leq$$

$$\leq C \max_{x \leq L} |R(x, ik_n)| \int_{-\infty}^{L} \int_{-\infty}^{x} |u(y)| \, dy \, dx =$$

$$= C\left[x \int_{-\infty}^{x} |u(y)| \, dx\right]_{-\infty}^{L} - C \int_{-\infty}^{L} x |u(x)| \, dx \; .$$

Now, if u(x) = [1], then:

$$\lim_{x\to-\infty} x \int_{-\infty}^{x} |u(y)|\,dy = \lim_{x\to-\infty} x^2|u(x)| = 0 .$$

So, ${}_{-\infty}\int^{L} f_n(x)dx < \infty$ and convergence of ${}_{-\infty}\int^{0} (\tilde{\phi}_n\tilde{\psi}_n - 1)dx$ is proved.

${}_0\int^{\infty} (\tilde{\phi}_n\tilde{\psi}_n + 1)dx$ can be proved to converge in exactly the same way as the proof that is given above. However, instead of working with $\tilde{\psi}_n(x) = R(x,ik_n)e^{-k_n x}$, we must use:

$$\tilde{\phi}_n(x)\tilde{\psi}_n(x) = \tilde{\psi}_\ell(x,ik_n)\psi_\ell(x,ik_n) ,$$

where

$$\psi_\ell(x,ik_n) = L(x,ik_n)e^{k_n x}$$

and L satisfies

$$L(x,k) = 1 + \int_x^{\infty} \frac{u(y)}{2ik} \{e^{2ik(y-x)} - 1\}L(y,k)dy , \quad k \in \bar{\mathbb{C}}^+ \setminus \{0\} .$$

$L(x,k)$ is continuous in (x,k) on $\mathbb{R} \times \bar{\mathbb{C}}^+$

$\tilde{\psi}_\ell(x,ik_n)$ is the eigenfunction for $\lambda = -k_n^2$, defined by:

$$\lim_{x\to\infty} \tilde{\psi}_\ell(x,ik_n)\psi_\ell(x,ik_n) = -1 .$$

Q.E.D.

Note that we have shown that:

(B.2.8) $$\int_{-\infty}^{0} |\tilde{\phi}_n\tilde{\psi}_n - 1|\,dx \quad \text{and} \quad \int_0^{\infty} |\tilde{\phi}_n\tilde{\psi}_n + 1|\,dx \quad \text{converge.}$$

Lemma (B.2.2):

If u(x) = [1] *and* f(u(x)) = [0], *then all the integrals occurring in* $H_n(t)$, *as defined in* (3.1.11,12), *converge.*

Proof:

From (2.2.30), we know that

(B.2.9) $$|\tilde{\psi}_n(x)| = |R(x,ik_n)|e^{k_n x} \leqq Ce^{k_n x}, \quad x < 0 ;$$

$$|\tilde{\psi}_n(x)| = |\tilde{c}_n \psi_\ell(x,ik_n)| = \tilde{c}_n |L(x,ik_n)| e^{-k_n x} \leqq Ce^{-k_n x}, \quad x > 0 .$$

Firstly, we will show convergence of:

$$I_1 := \int_{-\infty}^{\infty} \phi_n \psi_n \left\{ \left(\int_{-\infty}^{\infty} f(u)\psi_n^2 dx \right) \int_{-\infty}^{x} \psi_n^2 dx' - \int_{-\infty}^{x} f(u)\psi_n^2 dx' \right\} dx .$$

Using (B.2.9) and $\int_{-\infty}^{\infty} \psi_n^2(x)dx = 1$, we get:

$$|I_1| \leqq \left| \int_{-\infty}^{0} \phi_n \psi_n \left\{ \left(\int_{-\infty}^{\infty} f(u)\psi_n^2 dx \right) \int_{-\infty}^{x} \psi_n^2 dx' - \int_{-\infty}^{x} f(u)\psi_n^2 dx' \right\} dx \right| +$$

$$+ \left| \int_{0}^{\infty} \phi_n \psi_n \left\{ \int_{x}^{\infty} f(u)\psi_n^2 dx' - \left(\int_{-\infty}^{\infty} f(u)\psi_n^2 dx \right) \int_{x}^{\infty} \psi_n^2 dx' \right\} dx \right| \leqq$$

$$\leqq C\left\{ \int_{-\infty}^{0} \left(\int_{-\infty}^{x} e^{2k_n x'} dx' \right) dx + \int_{0}^{\infty} \left(\int_{x}^{\infty} e^{-2k_n x'} dx' \right) dx \right\} < \infty .$$

Secondly, we use (B.2.8) and $\phi_n \psi_n = \tilde{\phi}_n \tilde{\psi}_n$ to show convergence of:

$$I_2 := \int_{-\infty}^{\infty} \psi_n^2 \left\{ x + \int_{-\infty}^{x} (\phi_n \psi_n - 1) dx' \right\} dx$$

$$|I_2| = \left| \int_{-\infty}^{\infty} \psi_n^2 \left\{ - x + \int_{-\infty}^{0} (\phi_n \psi_n - 1) dx' + \int_{0}^{x} (\phi_n \psi_n + 1) dx' \right\} dx \right| \leqq$$

$$\leqq \int_{-\infty}^{\infty} \psi_n^2 \left\{ |x| + \int_{-\infty}^{0} |\phi_n \psi_n - 1| dx' + \int_{0}^{\infty} |\phi_n \psi_n + 1| dx' \right\} dx < \infty .$$

Convergence of the remaining integrals in $H_n(t)$ is trivial.

APPENDIX C

Proof that (5.1.10) can be replaced by the conditions (5.1.24) and (5.1.25)

Define $g(k,\tau)$ as:

$$g(k,\tau) = \frac{1}{k} \int_{-\infty}^{\infty} f(u(x,\tau))\psi^2(x,k,\tau)dx \ .$$

Suppose $W(0,\tau) = 0$ for $\tau \in [a_1,a_2] \subset [0,A]$.
Because of (5.1.24) we have:

$$\left| \int_{\substack{k=re^{i\varphi} \\ 0\leq\varphi\leq\pi}} \int_{a_1}^{a_2} g(k,\tau')\, e^{8ik^3\frac{(\tau-\tau')}{\delta}} e^{2ik(x+y)}\, d\tau'\, dk \right| =$$

$$= \left| \int_{a_1}^{a_2} \int_{\substack{k=re^{i\varphi} \\ 0\leq\varphi\leq\pi}} g(k,\tau')\, e^{8ik^3\frac{(\tau-\tau')}{\delta}} e^{2ik(x+y)}\, dk\, d\tau' \right| =$$

$$= \left| \pi i \int_{a_1}^{a_2} \int_{-\infty}^{\infty} f(u(x,\tau'))\psi^2(x,0,\tau')\, dx\, d\tau' \right| \leq C(a_2 - a_1) \ .$$

So:
Let $I_i \subset [0,A]$, $i = 1,\ldots,p$, be disjoint intervals on which $W(0,\tau) = 0$, and $W(0,\tau) \neq 0$ for $\tau \in [0,A] \setminus \bigcup_i I_i$.
Then:

$$\left| \int_{\substack{k=re^{i\varphi} \\ 0\leq\varphi\leq\pi}} \int_{\bigcup_i I_i} g(k,\tau')\, e^{8ik^3\frac{(\tau-\tau')}{\delta}} e^{2ik(x+y)}\, d\tau'\, dk \right| \leq C \sum_i \mu(I_i) \ , \tag{C.1}$$

where $\mu(I_i)$ is the Lebesgue measure of I_i.
Now, for estimating $\int_{-\infty}^{\infty} II(k,\tau)dk$, for $\tau' \in [0,\tau] \setminus \bigcup_i I_i$, we integrate along the rectangle $\{k \in \mathbb{C} \mid 0 \leq \mathrm{Im}\ k \leq \eta,\ |\mathrm{Re}\ k| \leq \rho\}$ and take the limit

$\rho \to \infty$, while for $\tau' \in [0,\tau] \cap \bigcup_i I_i$, we integrate along the rectangle minus the semicircle with radius r: $\{k \in \mathbb{C} \mid 0 \leqq \text{Im } k \leqq \eta, |\text{Re } k| \leqq \rho, |k| \geqq r\}$ and take the limits $\rho \to \infty$, $r \to 0$.

From (C.1) it immediately follows that, if $W(0,\tau) \neq 0$ almost everywhere on $[0,A]$, then, no extra contribution will come from the integration along the semicricle.

This proves the statement.

Q.E.D.

APPENDIX D

Proofs of Lemmas (6.1) and (6.2)

i) Proof of Lemma (6.1)

For each $\tau \in [0,A]$ fixed, we have:

$$\int_{-\infty}^{\infty} f(u_s(x,\tau))\psi_{ms}^2(x,\tau)dx = \int_{-\infty}^{\infty} f(\bar{u}_s(z_m,\tau)\bar{\psi}_{ms}^2(z_m,\tau)dz_m ,$$

where $\bar{u}_s(z_m,\tau) = u_s(x,\tau)$.
We split the integration interval into three parts:

$$\text{I:} \quad -\infty < z_m \leq \tfrac{1}{2}(\varphi_{m-1} - \varphi_m) ,$$

$$\text{II:} \quad \tfrac{1}{2}(\varphi_{m-1} - \varphi_m) \leq z_m \leq \tfrac{1}{2}(\varphi_{m+1} - \varphi_m) ,$$

$$\text{III:} \quad \tfrac{1}{2}(\varphi_{m+1} - \varphi_m) \leq z_m < \infty .$$

Now, using (3.2.16) and (3.2.13), we get:

$$\left| \int_{-\infty}^{\frac{1}{2}(\varphi_{m-1}-\varphi_m)} f(\bar{u}_s)\bar{\psi}_{ms}^2\, dz_m \right| \leq C \int_{-\infty}^{\frac{1}{2}(\varphi_{m-1}-\varphi_m)} e^{2k_m z_m} dz_m =$$

$$= C e^{k_m(\varphi_{m-1}-\varphi_m)} = O\left(\exp -\frac{\alpha\tau}{\delta(\varepsilon)}\right) , \quad \text{for some positive constant } \alpha .$$

And analogously:

$$\int_{III} f(\bar{u}_s)\bar{\psi}_{ms}^2\, dz_m = O\left(\exp -\frac{\alpha\tau}{\delta(\varepsilon)}\right) ,$$

$$\int_{I} f\left(-2k_m^2 \operatorname{sech}^2 k_m(z_m - \delta_m^+)\right)\cdot \tfrac{1}{2}k_m \operatorname{sech}^2 k_m(z_m - \delta_m^+)dz_m = O\left(\exp -\frac{\alpha\tau}{\delta(\varepsilon)}\right) ,$$

$$\int_{III} f\left(-2k_m^2 \operatorname{sech}^2 k_m(z_m-\delta_m^+)\right)\cdot \tfrac{1}{2}k_m \operatorname{sech}^2 k_m(z_m-\delta_m^+)dz_m = 0\left(\exp -\frac{\alpha\tau}{\delta(\varepsilon)}\right) .$$

For the region II, we use $u_s = -4\sum_{n=1}^{N} k_n \psi_{ns}^2$, (3.2.15,16) and the special structure of the perturbation to get:

$$f(\bar{u}_s) = f(-4k_m \bar{\psi}_{ms}^2) + 0\left(\exp -\frac{\alpha\tau}{\delta(\varepsilon)}\right) .$$

And so, it follows that:

$$\text{(D.1)} \qquad \int_{II} f(\bar{u}_s)\bar{\psi}_{ms}^2\, dz_m = \int_{II} \left(f(-4k_m \bar{\psi}_{ms}^2) + 0\left(e^{-\frac{\alpha\tau}{\delta(\varepsilon)}}\right)\right)\bar{\psi}_{ms}^2\, dz_m .$$

Using (3.2.18), we see:

$$\text{(D.2)} \qquad f(-4k_m \bar{\psi}_{ms}^2) = f\left(\left(1+0(e^{-\frac{\alpha\tau}{\delta(\varepsilon)}})\right)\cdot\left(-2k_m^2 \operatorname{sech}^2 k_m(z_m-\delta_m^+)\right)\right) =$$

$$= f\left(-2k_m^2 \operatorname{sech}^2 k_m(z_m-\delta_m^+)\right) + 0\left(e^{-\frac{\alpha\tau}{\delta(\varepsilon)}}\right) f_0\left(-2k_m^2 \operatorname{sech}^2 k_m(z_m-\delta_m^+)\right) ,$$

where f_0 is an operator of the same form as f.
Combining (D.1) and (D.2) we get:

$$\int_{II} f(\bar{u}_s)\bar{\psi}_{ms}^2 dz_m = \int_{II} f\left(-2k_m^2 \operatorname{sech}^2 k_m(z_m-\delta_m^+)\right)\cdot \tfrac{1}{2}k_m \operatorname{sech}^2 k_m(z_m-\delta_m^+)dz_m + 0\left(e^{-\frac{\alpha\tau}{\delta(\varepsilon)}}\right) .$$

Finally, combining the results for the regions I, II and III, we obtain the estimate required.

Q.E.D.

ii) Proof of Lemma (6.2):

We start by defining:

$\bar{x} = x - v\tau\delta^{-1}(\varepsilon)$; $\bar{u}(\bar{x},\tau) = u(x,\tau)$, etc.;

$\tau_m = m\delta(\varepsilon)\log\frac{1}{\varepsilon}$;

$D = \mathbb{R} \times [\tau_m, A]$;

$D^- = (-\infty, M] \times [\tau_m, A]$, $D^+ = [M, \infty) \times [\tau_m, A]$.

On the region D^-, we can use the following estimate (use (3.2.9), Lemma (3.2.1) and $\psi_n = d_n \psi_r(x, ik_n, \tau)$):

(D.3) $$|\bar{\psi}_n(\bar{x},\tau)| \;,\; |\frac{\partial}{\partial \bar{x}} \bar{\psi}_n(\bar{x},\tau)| \;,\; |\bar{\psi}_{ns}(\bar{x},\tau)| \;,\; |\frac{\partial}{\partial \bar{x}} \bar{\psi}_{ns}(\bar{x},\tau)|$$

$$\leq C\, d_n(0) \exp \frac{k_n}{\delta(\varepsilon)} \left\{ v\tau - \int_0^\tau 4k_n^2\, d\tau' + O(\varepsilon) \right\} e^{k_n \bar{x}}$$

$$\leq C\, e^{-\frac{\alpha\tau}{\delta(\varepsilon)}}\, e^{k_n \bar{x}} \quad (= O(\varepsilon^{\alpha m}) \text{ uniformly on } D^-) \;,$$

where we take $v < 4M_1^2$ and $\alpha := k_n(4M_1^2 - v)$. ($M_1$ defined as in (3.2.3b.)

From (D.3) it is easy to see that we also have:

(D.4) $$\int_{-\infty}^{M} |w(\bar{x},\tau)|\, d\bar{x} \leq C\, e^{-\frac{\alpha\tau}{\delta(\varepsilon)}} \;, \quad \text{on } [\tau_m, A] \;,$$

$$\text{for } \; w(\bar{x},\tau) \in \left\{ \bar{\psi}_n(\bar{x},\tau) \,,\, \frac{\partial}{\partial \bar{x}} \bar{\psi}_n(\bar{x},\tau) \,,\, \bar{\psi}_{ns}(\bar{x},\tau) \,,\, \frac{\partial}{\partial \bar{x}} \psi_{ns}(\bar{x},\tau) \right\} .$$

On the region D^+, we use condition (6.4). Moreover, we use that:

(D.5) $$\psi_n \;;\; \frac{\partial}{\partial x}\psi_n \;;\; \psi_{ns} \;;\; \frac{\partial}{\partial x}\psi_{ns} \;;\; \frac{\partial^s u}{\partial x^s} \text{ and } \frac{\partial^s u_s}{\partial x^s} \;, \quad s = 0, 1, \ldots, 2j \;,$$

are uniformly bounded on D,

and

(D.6) $$\int_{-\infty}^{\infty} f(u)\psi_n^2 - f(u_s)\psi_{ns}^2\, dx = O(\zeta(\varepsilon)) \;, \quad \text{uniformly on } [\tau_m, A] \;,$$

for some order-function $\zeta(\varepsilon)$,

implies that:

$$\int_{-\infty}^{\infty} |f(u)\psi_n^2 - f(u_s)\psi_{ns}^2|\, dx = O(\zeta(\varepsilon)) \;, \quad \text{uniformly on } [\tau_m, A] \;.$$

(Of course, in general, this implication is not true. However, in this case we already know that both of the integrals converge and that the smallness of $_{-\infty}\int^{\infty} f(u)\psi_n^2 - f(u_s)\psi_{ns}^2\, dx$ is due to the smallness of the integrand on $[\tau_{-}, A]$.)

The proof is based on an induction procedure that consists of the following steps:

1°. Show that the lemma holds for $f(u) = L(u)$.

2°. Show that, if the lemma holds for $f(u) = f^o(u)$ (where $f^o(u)$ is of structure (6.2)), then it holds for $f(u) = f^o(u)(\partial^s u/\partial x^s)$, $0 \leq s \leq j$, too.

3°. Show that the lemma holds for $h(u) = f(u) + g(u)$, when it holds for $f(u)$ and $g(u)$.

Ad 1°:

$$\int_{-\infty}^{\infty} |L(\bar{u})\bar{\psi}_n^2 - L(\bar{u}_s)\bar{\psi}_{ns}^2|\,d\bar{x} \leq$$

$$\leq \int_M^{\infty} |\tfrac{1}{2}(L(\bar{u}) - L(\bar{u}_s))(\bar{\psi}_n^2 + \bar{\psi}_{ns}^2) + \tfrac{1}{2}(L(\bar{u}) + L(\bar{u}_s))(\bar{\psi}_n^2 - \bar{\psi}_{ns}^2)|\,d\bar{x} +$$

$$+ \int_{-\infty}^{M} \left(|L(\bar{u})|\,\bar{\psi}_n^2 + |L(\bar{u}_s)|\,\bar{\psi}_{ns}^2\right) d\bar{x} \leq$$

$$\leq C \sup_{D^+} |\bar{u} - \bar{u}_s| + C \sup_{D^+} |\bar{\psi}_n - \bar{\psi}_{ns}| + C\, e^{-2\alpha \frac{\tau}{\delta(\varepsilon)}} =$$

$$= O\left(q(\varepsilon) + e^{-2\alpha \frac{\tau}{\delta(\varepsilon)}}\right), \quad \tau \in [\tau_m, A]\ .$$

Ad 2°:

We know that:

$$\| f^o(\bar{u})\bar{\psi}_n^2 - f^o(\bar{u}_s)\bar{\psi}_{ns}^2 \|_{L_1} = O\left(q(\varepsilon) + e^{-2\alpha \frac{\tau}{\delta(\varepsilon)}}\right).$$

Now, for $f(\bar{u}) = f^o(\bar{u})(\partial^s \bar{u}/\partial \bar{x}^s)$, we have:

$$\left| \int_{-\infty}^{\infty} \left(f(\bar{u})\bar{\psi}_n^2 - f(\bar{u}_s)\bar{\psi}_{ns}^2\right) dx \right| \leq$$

$$\leq \tfrac{1}{2} \left| \int_{-\infty}^{\infty} \left(f^o(\bar{u})\bar{\psi}_n^2 - f^o(\bar{u}_s)\bar{\psi}_{ns}^2\right) \left(\frac{\partial^s \bar{u}}{\partial \bar{x}^s} + \frac{\partial^s \bar{u}_s}{\partial \bar{x}^s}\right) d\bar{x} \right| +$$

$$+ \tfrac{1}{2} \left| \int_{-\infty}^{\infty} \left(f^o(\bar{u})\bar{\psi}_n^2 + f^o(\bar{u}_s)\bar{\psi}_{ns}^2\right) \left(\frac{\partial^s\bar{u}}{\partial\bar{x}^s} - \frac{\partial^s\bar{u}_s}{\partial\bar{x}^s}\right) d\bar{x} \right| := I + II .$$

$$I \leq \tfrac{1}{2} \sup_D \left| \frac{\partial^s\bar{u}}{\partial\bar{x}^s} + \frac{\partial^s\bar{u}_s}{\partial\bar{x}^s} \right| \cdot \| f^o(\bar{u})\bar{\psi}_n^2 - f^o(\bar{u}_s)\bar{\psi}_{ns}^2 \|_{L_1} =$$

$$= O\left(q(\varepsilon) + e^{-2\alpha\frac{\tau}{\delta(\varepsilon)}}\right) .$$

Performing partial integration s-times and using that $\bar{\psi}_n$ and $\bar{\psi}_{ns}$ are L_2-solutions of the Schrödinger equation, we get:

$$II = \tfrac{1}{2} \left| \int_{-\infty}^{\infty} \frac{\partial^s}{\partial\bar{x}^s} [f^0(\bar{u})\bar{\psi}_n^2 + f^0(\bar{u}_s)\bar{\psi}_{ns}^2] \cdot (\bar{u} - \bar{u}_s) d\bar{x} \right| =$$

$$= \left| \int_{-\infty}^{\infty} \left(P_1 \bar{\psi}_n^2 + P_2 \bar{\psi}_n \frac{\partial\bar{\psi}n}{\partial\bar{x}} + P_3 \bar{\psi}_{ns}^2 + P_4 \bar{\psi}_{ns} \frac{\partial\bar{\psi}_{ns}}{\partial\bar{x}}\right) (\bar{u} - u_s) d\bar{x} \right| ,$$

where: P_1, P_2 are polynomials in $\bar{u}$ and $\bar{x}$-derivatives up to degree 2s of $\bar{u}$, with multiples of $\bar{u}$-derivatives up to degree s of $L(\bar{u})$ as coefficients,

while: P_3, P_4 are polynomials in $\bar{u}_s$ and $\bar{x}$-derivatives up to degree 2s of $\bar{u}_s$, with multiples of $\bar{u}_s$-derivatives up to degree s of $L(\bar{u}_s)$ as coefficients.

Using (6.4), (D.5) and the boundedness of the $\bar{u}$-derivatives of $L(\bar{u})$, respectively, of the $\bar{u}_s$-derivatives of $L(\bar{u}_s)$, it is now easily seen that:

$$II = O\left(q(\varepsilon) + e^{-2\alpha\frac{\tau}{\delta(\varepsilon)}}\right) .$$

<u>Ad 3°:</u>

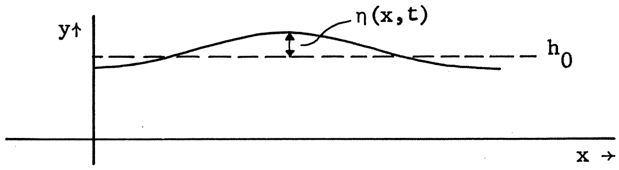
Trivial.

Q.E.D.

APPENDIX E

E.1. Derivation of the KdV-equation for shallow-water waves

We consider a two-dimensional model for the flow of an incompressible, irrotational, non-viscous fluid in a canal with a flat horizontal bottom (see picture).



We define:

(E.1.1) h_0: the depth of the canal with water at rest,

φ : the velocity field potential $(\vec{v} = \nabla\varphi)$,

q : the gravitational acceleration.

The system is described by the following set of equations:

(E.1.2) a) $\Delta\varphi = 0$, $0 < y < h_0 + \eta(x,t))$. (Conservation of mass)

b) $\varphi_t + \frac{1}{2}(\varphi_x^2 + \varphi_y^2) + g\eta = 0$, $y = h_0 + \eta(x,t)$. (Equation of motion)

c) $\varphi_y = 0$, $y = 0$. (Boundary condition at $y = 0$)

d) $\eta_t + \varphi_x\eta_x = \varphi_y$, $y = h_0 + \eta(x,t)$. (Equation of motion for the free boundary $y = h_0 + \eta(x,t)$. Surface tension is neglected)

We are looking for waves with the following properties:

(E.1.3) $\alpha = \frac{a}{h_0} << 1$, a is a typical wave amplitude,

$$\beta = \frac{h_0^2}{\ell^2} << 1 \ , \quad \ell \text{ is a typical wave length.}$$

We introduce the following set of dimensionless variables (the old variables are given a bar):

(E.1.4) $$x = \frac{1}{\ell}(\bar{x} - c_0\bar{t}) \ , \quad y = \frac{1}{h_0}\bar{y} \ , \quad t = \frac{\alpha c_0}{\ell}\bar{t} \ ,$$

$$\eta(x,t) = \frac{1}{a}\bar{\eta}(\bar{x},\bar{t}) \ , \quad \varphi(x,y,t) = \frac{c_0}{g\ell a}\bar{\varphi}(\bar{x},\bar{y},\bar{t}) \ .$$

Here $c_0 = \sqrt{gh_0}$ is the phase-velocity of nondispersive gravity waves. (See [W], § 13.3,4.)

In these new variables, we have the following set of equations:

(E.1.5)

a) $\beta\varphi_{xx} + \varphi_{yy} = 0 \ ,$ $\qquad 0 < y < 1+\alpha\eta \ ,$

b) $\varphi_y = 0 \ ,$ $\qquad y = 0 \ ,$

c) $\left(\alpha\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\eta + \alpha\varphi_x\eta_x - \frac{1}{\beta}\varphi_y = 0 \ ,$ $\qquad y = 1+\alpha\eta \ ,$

d) $\left(\alpha\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\varphi + \frac{1}{2}\alpha\varphi_x^2 + \frac{\alpha}{2\beta}\varphi_y^2 + \eta = 0 \ ,$ $\qquad y = 1+\alpha\eta \ .$

We search for a solution of (E.1.5a,b) of the form:

(E.1.6) $$\varphi(x,y,t) = \sum_{n=0}^{\infty} y^n f_n(x,t) \ .$$

Substitution of (E.1.6) into (E.1.5a) and using (E.1.5b), leads to:

(E.1.7) $$\varphi(x,y,t) = \sum_{n=0}^{\infty} \frac{(-\beta)^n y^{2n}}{(2n)!} \frac{\partial^{2n}}{\partial x^{2n}} f(x,t) \ , \quad \text{with } f(x,t) = f_0(x,t) \ .$$

Next, we substitute (E.1.7) into (E.1.5c) and (E.1.5d), respectively, and then differentiate to x. This gives:

(E.1.8) $$-\eta_x + w_x + \alpha(\eta_t + w\eta_x + \eta w_x) + \beta(-\frac{1}{6}w_{xxx}) +$$
$$-\frac{1}{2}\alpha\beta(\eta_x w_{xx} + \eta w_{xxx}) + \frac{1}{120}\beta^2 w_{xxxxx} = O(\alpha^3 + \beta^3) \ .$$

$$(E.1.9)\quad \eta_x - w_x + \alpha(w_t + ww_x) + \frac{1}{2}\beta w_{xxx} +$$

$$-\frac{1}{2}\alpha\beta(ww_{xxx} + w_{xxt} - w_x w_{xx} - 2\eta_x w_{xx} - 2\eta w_{xxx}) - \frac{1}{24}\beta^2 w_{xxxxx} = O(\alpha^3 + \beta^3) \ .$$

In the above equations, $w(x,t) = \frac{\partial}{\partial x} f(x,t)$ is the first term in the expansion of the horizontal velocity $\frac{\partial}{\partial x}\varphi(x,t)$.
Equations (E.1.8), as well as (E.1.9), imply that a first order approximation will give $\eta_x = w_x$, so that:

$$(E.1.10)\quad \eta(x,t) = w(x,t) + q(t) = \varphi_x(x,t) + q(t) + O(\alpha + \beta) \ .$$

From a physical point of view, it is quite unlikely that the vertical displacement $\eta(x,t)$ and the horizontal velocity $\varphi_x(x,t)$ differ by a function that is only t-dependent. Therefore, we take $q(t) \equiv 0$.
We now specify the terms in the expansion of $\eta(x,t)$ up to order $O(\alpha^2 + \beta^2)$:

$$(E.1.11)\quad \eta(x,t) = w(x,t) + \alpha A(x,t) + \beta B(x,t) + \alpha^2 C(x,t) + \alpha\beta D(x,t) +$$

$$+ \beta^2 E(x,t) + O(\alpha^3 + \beta^3) \ .$$

For the same reason as mentioned above, the functions A and B neither contain constants nor parts depending only on t.

Substituting (E.1.11) into (E.1.8) and (E.1.9), respectively, we obtain:

$$(E.1.12)\quad \alpha(\eta_t + 2\eta\eta_x + A_x) + \beta(B_x - \frac{1}{6}\eta_{xxx}) + \alpha^2(C_x + \eta A_x + \eta_x A) +$$

$$+ \alpha\beta(D_x + \eta B_x + B\eta_x - \frac{1}{6}A_{xxx} - \frac{1}{2}\eta_x\eta_{xx} - \frac{1}{2}\eta\eta_{xxx}) +$$

$$+ \beta^2(E_x - \frac{1}{6}B_{xxx} + \frac{1}{120}\eta_{xxxxx}) = O(\alpha^3 + \beta^3) \ ;$$

$$(E.1.13)\quad \alpha(\eta_t + \eta\eta_x - A_x) + \beta(-B_x + \frac{1}{2}\eta_{xxx}) + \alpha^2(-C_x + \eta A_x + A\eta_x + A_t) +$$

$$+ \alpha\beta(-D_x + \eta B_x + B\eta_x + B_t - \frac{1}{2}\eta_{xxt} + \frac{3}{2}\eta_x\eta_{xx} + \frac{1}{2}\eta\eta_{xxx} + \frac{1}{2}A_{xxx}) +$$

$$+ \beta^2(\frac{1}{2}B_{xxx} - \frac{1}{24}\eta_{xxxxx} - E_x) = O(\alpha^3 + \beta^3) \ .$$

Adding and subtracting the equations (E.1.12) and (E.1.13) leads to:

(E.1.14) $\alpha(2\eta_t + 3\eta\eta_x) + \frac{1}{3}\beta\eta_{xxx} + \alpha^2(2\eta A_x + 2A\eta_x + A_t) +$

$+ \alpha\beta(2\eta B_x + 2B\eta_x + \frac{1}{3}A_{xxx} + \eta_x\eta_{xx} + B_t - \frac{1}{2}\eta_{xxt}) +$

$+ \beta^2(\frac{1}{3}B_{xxx} - \frac{1}{30}\eta_{xxxxx}) = O(\alpha^3 + \beta^3)$.

(E.1.15) $\alpha(2A_x + \eta\eta_x) + \beta(2B_x - \frac{2}{3}\eta_{xxx}) = O(\alpha^2 + \beta^2)$.

From (E.1.15) we can see that:

(E.1.16) a) $2A_x + \eta\eta_x = 0$, and consequently: $A(x,t) = -\frac{1}{4}\eta^2$,

so: $A_t = -\frac{1}{2}\eta\eta_t$;

b) $2B_x - \frac{2}{3}\eta_{xxx} = 0$, and consequently: $B(x,t) = \frac{1}{3}\eta_{xx}$,

so: $B_t = \frac{1}{3}\eta_{xxt}$.

From (E.1.14) we see that:

(E.1.17) $\eta_t = -\frac{3}{2}\eta\eta_x - \frac{1}{6}\beta\alpha^{-1}\eta_{xxx} + O(\alpha + \beta + \beta^2\alpha^{-1})$.

Substituting (E.1.17) in the expressions for A_t and B_t in (E.1.16) leads to:

(E.1.18) a) $\alpha^2 A_t = \frac{3}{4}\alpha^2\eta^2\eta_x + \frac{1}{12}\alpha\beta\eta\eta_{xxx} + O(\alpha^3 + \beta^3)$;

b) $\alpha\beta(B_t - \frac{1}{2}\eta_{xxt}) = -\frac{1}{6}\alpha\beta\eta_{xxt} =$

$= \frac{1}{4}\alpha\beta(3\eta_x\eta_{xx} + \eta_{xxx}) + \frac{1}{36}\beta^2\eta_{xxxxx} + O(\alpha^3 + \beta^3)$.

Finally substituting (E.1.16,17,18) into (E.1.14) will give:

(E.1.19) $\alpha(2\eta_t + 3\eta\eta_x) + \frac{1}{3}\beta\eta_{xxx} - \frac{3}{4}\alpha^2\eta^2\eta_x + \alpha\beta(\frac{5}{6}\eta\eta_{xxx} + \frac{23}{12}\eta_x\eta_{xx}) +$

$+ \beta^2\frac{19}{180}\eta_{xxxxx} = O(\alpha^3 + \beta^3)$.

Now, we consider the case in which α and β are of the same order of magnitude:

(E.1.20) $\alpha = \varepsilon \ ; \quad \beta = C\varepsilon \ .$

Moreover, in order to get the KdV-equation in its most familiar form, we need to introduce new variables:

$$\text{(E.1.21)} \quad \tilde{x} = \left(\frac{3}{2c}\right)^{\frac{1}{2}} x \ ; \quad \tilde{t} = \frac{1}{4}\left(\frac{3}{2c}\right)^{\frac{1}{2}} t \ ; \quad u(\tilde{x},\tilde{t}) = -\eta(x,t) \ .$$

For convenience, we will omit the $\sim$'s and get:

$$\text{(E.1.22)} \quad u_t - 6uu_x + u_{xxx} = \varepsilon\{\tfrac{3}{2}u^2u_x + \tfrac{5}{2}uu_{xxx} + \tfrac{23}{4}u_xu_{xx} - \tfrac{19}{40}u_{xxxxx}\} + O(\varepsilon^2) \ .$$

This is the KdV-equation + first order terms, such as used in § VII.3.

We will now show that the physical equivalent of a solitary wave solution $u(x,t) = -2\kappa^2 \operatorname{sech}^2 \kappa(x - 4\kappa^2 t)$, of the KdV-equation, is a shallow waterwave. Transforming back to physical coordinates: η, x_f, t_f, we get:

$$\eta(x_f,t_f) = 2a\kappa^2 \operatorname{sech}^2 \frac{\kappa}{\ell}\sqrt{\frac{3}{2c}}\,(x_f - c_0(1+\alpha\kappa^2)t_f) \ .$$

Inserting $h_0 = 1$ and $\alpha = a = 1/\ell^2 \ll 1$, we get:

$$\eta(x_f,t_f) = 2\,\frac{\kappa^2}{\ell^2}\operatorname{sech}^2 \frac{\kappa}{\ell}\sqrt{\frac{3}{2}}\left(x_f - c_0\left(1+\frac{\kappa^2}{\ell^2}\right)t_f\right) \ .$$

Indeed, this represents a shallow-waterwave.
Note that the wave-velocity $c_0(1+\kappa^2/\ell^2)$ is an $O(\varepsilon)$-perturbation of the phase-velocity c_0 of nondispersive gravity waves.

E.2. Solving (7.4.6)

In this section, we will solve:

$$\text{(E.2.1)} \quad \begin{cases} u_{xx} + (12\kappa^2 \operatorname{ch}^{-2}\kappa x - 4\kappa^2)u = \kappa^6\{2u\operatorname{ch}^{-6}\kappa x - 48\operatorname{ch}^{-4}\kappa x + \\ \qquad\qquad + [\frac{76}{5} + 2a]\operatorname{ch}^{-2}\kappa x\} \ ; \\ \lim\limits_{|x|\to\infty} u(x) = 0 \ . \end{cases}$$

We use the notation:

(E.2.2) $\mathrm{ch} \equiv \cosh$; $\mathrm{sh} \equiv \sinh$; $\kappa x = y$; $\bar{u}(y) = u(x)$.

Define $L: C^2(\mathbb{R}) \to C(\mathbb{R})$ by

(E.2.3) $(Lf)(y) = (D^2 f)(y) + (12\,\mathrm{ch}^{-2} y - 4) f(y)$.

(E.2.1) is equivalent to:

(E.2.4) $(L\bar{u})(y) = \kappa^4 \{24\,\mathrm{ch}^{-6} y - 48\,\mathrm{ch}^{-4} y + [\frac{76}{5} + 2a]\,\mathrm{ch}^{-2} y\}$.

It is easily seen that a solution of:

(E.2.5) $(Lw)(y) = \kappa^4 \{24\,\mathrm{ch}^{-6} y - 48\,\mathrm{ch}^{-4} y\}$

is given by:

(E.2.6) $w(y) = -\kappa^4 \{2\,\mathrm{ch}^{-2} y - 3\,\mathrm{ch}^{-4} y\}$.

So, what we need is a solution of:

(E.2.7) a) $(Lv)(y) = \kappa^4 (\frac{76}{5} + 2a)\,\mathrm{ch}^{-2} y$;

b) $\lim_{|y| \to \infty} v(y) = 0$.

We first solve the homogeneous equation

(E.2.8) $(L\psi)(y) = 0$.

From the theory in [F], we know that a solution of (E.2.8) is given by:

(E.2.9) $\psi_1(y) = \mathrm{ch}^4 y\ \mathrm{sh}\, y\ F(\frac{7}{2}, \frac{3}{2}, \frac{3}{2}, -\mathrm{sh}^2 y)$.

The hypergeometric function F satisfies:

$$F(a,b,b,z) = (1 - z)^{-a} ,$$

so that

(E.2.10) $\psi_1(y) = \mathrm{ch}^{-3} y\ \mathrm{sh}\, y$.

The general solution of (E.2.8) can now be found by using order-reduction. We introduce: $\psi(y) = \psi_1(y)\varphi(y)$, and we get:

$$\text{(E.2.11)} \quad \psi(y) = A\,\mathrm{ch}^{-3}y\ \mathrm{sh}\,y + B\,\mathrm{ch}^{-3}y\ \mathrm{sh}\,y\,(-32\coth y + \mathrm{sh}\,4y + 16\,\mathrm{sh}\,2y + 60\,y)\ .$$

The solution of the inhomogeneous equation (E.2.7a) can be calculated using the variation of constants method. We find:

$$\text{(E.2.12)} \quad v(y) = A\,\mathrm{ch}^{-3}y\ \mathrm{sh}\,y + B\,\mathrm{ch}^{-3}y\ \mathrm{sh}\,y\,(-32\coth y + \mathrm{sh}\,4y + 16\,\mathrm{sh}\,2y + 60\,y) + \kappa^4(\tfrac{19}{5} + \tfrac{1}{2}a)(\mathrm{ch}^{-2}y - y\,\mathrm{ch}^{-3}y\ \mathrm{sh}\,y)\ .$$

With (E.2.7b), we see that: $B = 0$, A is arbitrary.
So, the solution $u(x)$ of (E.2.1) is found to be given by:

$$\text{(E.2.13)} \quad u(x) = w(\kappa x) + v(\kappa x) = \kappa^4\{(\tfrac{9}{5} + \tfrac{1}{2}a)\,\mathrm{ch}^{-2}\kappa x - 3\,\mathrm{ch}^{-4}\kappa x - (\tfrac{19}{5} + \tfrac{1}{2}a)\,\kappa x\ \mathrm{ch}^{-3}\kappa x\ \mathrm{sh}\,\kappa x\} + A\,\mathrm{ch}^{-3}\kappa x\ \mathrm{sh}\,\kappa x\ .$$

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LIST OF SYMBOLS

Some of the symbols used in this tract are listed below.
We only refer to the pages where the symbols are defined, or occur for the first time.
The meaning of the symbols with an asterisk *, depends on the context in which they are used.
The list is presented in alphabetical order; first Latin alphabet, then Greek.

a	15
a_0	56
b	15
$\tilde{b}$	83
b_s	96
$\bar{C}_+$	13
$C(x)$	24
c_n	21
$\tilde{c}_n$	20
c_{mn}	25
d_n	21
E_n	41
E_n^0	112
$f(u)$	Perturbation on the KdV-equation
G	16
* G_n	16
* G_n	33
$\hat{G}_n$	17
$g(k)$	95,107
H_f	113
H_n	33
$\bar{H}_{1s}^o$	119
h_n	41
k_n	20
k_n^o	95
* L	13
* L	97
ℓ_+	14, 15
ℓ_-	14, 15
M_1	36
M_2	36
$[m]$	12
$[m]_u$	25
N	3, 20
p_n	97
p_n^o	112

$q(\varepsilon)$	108
R	13
R_n	35
r	67
r_s	96
r_+	14, 15
r_-	14, 15
*S	8, 27
*S	56
S_s	34
$s(\varepsilon)$	52
T	59
T_c	54
T_d	54
$U(x)$	3
U_0	17
$U_0(x)$	17
u	Solution of a (p)KdV; potential in the S.E.
u_c	3, 5, 34
u_d	67
u_s	3, 5, 29, 34
W	14, 15
$\hat{W}$	15
z_n	38, 40
$\tilde{z}_n$	40

$\alpha(k)$	20
β	27, 53
β_c	57
β_d	54
γ_n	21, 22
$\delta(\varepsilon)$	4
δ_n	22
δ_n^+	29, 40
δ_n^-	29
ε	small parameter
η	85
θ_n	32
κ_n	95
λ	5, 12
λ_n	20
μ_j	36
τ	36
τ_m	110
ϕ_n	21
$\tilde{\phi}_n$	32
φ_n	37
*ψ	5, 12
*ψ	15
ψ_ℓ	13
ψ_n	6, 21
$\tilde{\psi}_n$	20
ψ_r	13
ψ_s	96
ψ_{ns}	6, 29, 34
Ω	27, 53
Ω_c	27, 53
Ω_d	27, 53
Ω_{cs}	96

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